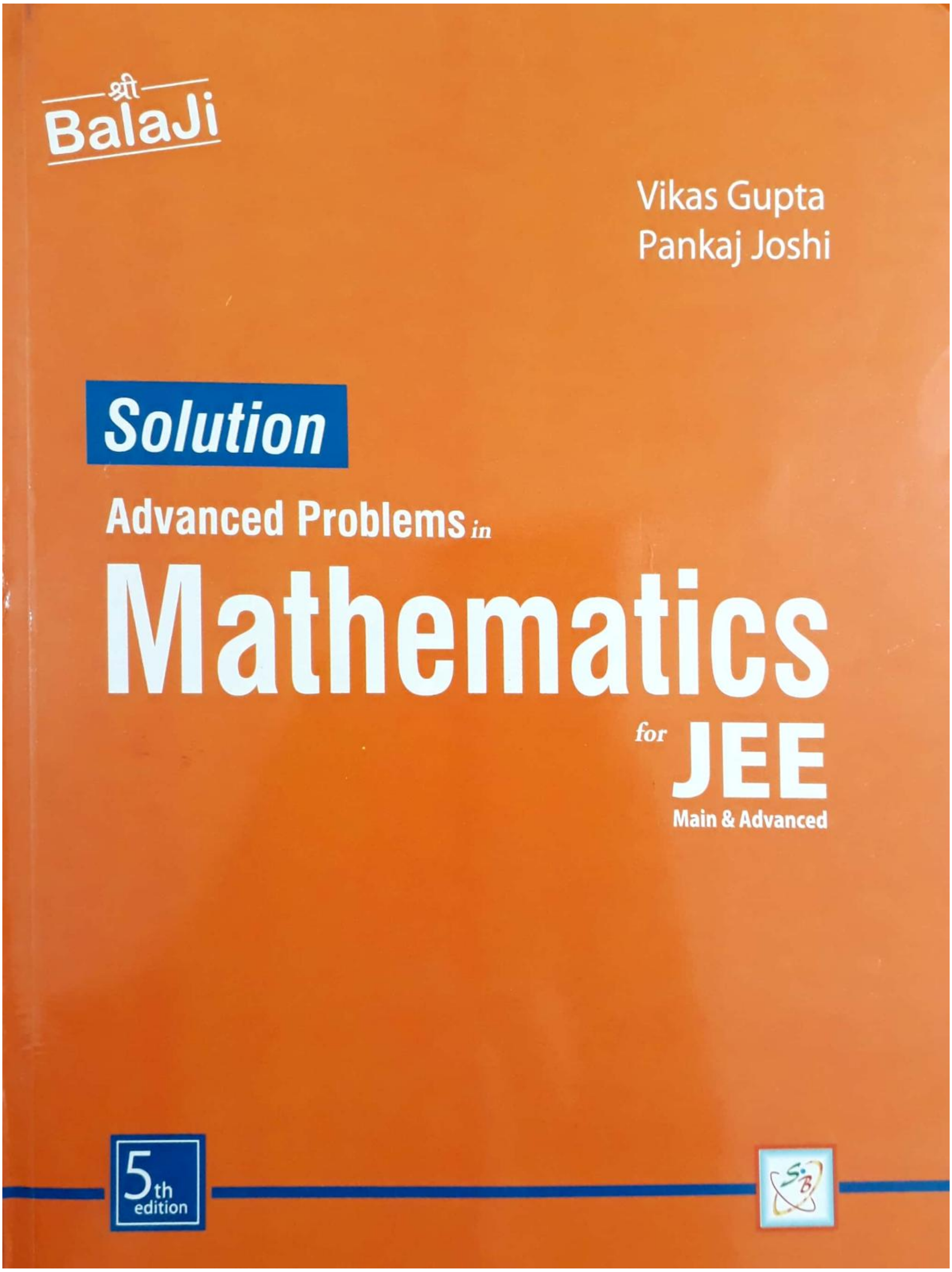
Balaji

Solution to Advanced Problems in Mathematics Chapter 1 to 8

for IIT JEE Main and Advanced

by

Vikas Gupta and Pankaj Joshi



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BalaJi

Vikas Gupta
Pankaj Joshi

Solution

Advanced Problems in Mathematics for JEE

Main & Advanced

5th edition

श्री
BalaJi

SOLUTION *to*
Advanced Problems
in
MATHEMATICS
for
JEE (MAIN & ADVANCED)

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Chapter 1 – Function

1

FUNCTION

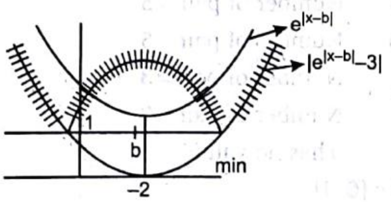
Exercise-1 : Single Choice Problems

1. $f(x) = \log_2(2 - 2\log_{\sqrt{2}}(16\sin^2 x + 1))$

$0 \le \log_{\sqrt{2}}(16\sin^2 x + 1) \le \log_2 17 \quad \Rightarrow 2 - 2\log_2 17 \le 2 - 2\log_{\sqrt{2}}(16\sin^2 x + 1) \le 2$

$\Rightarrow \quad 0 < 2 - 2\log_{\sqrt{2}}(16\sin^2 x + 1) \le 2 \quad \Rightarrow f(x) \le 1$

2. For any $b \in R$ $e^{|x-b|}$ is



$|e^{|x-b|} - a|$ has four distinct solutions $a > 3$ so $a \in (3, \infty)$

3. Domain $= [-1, 1]$ and both are increasing functions.

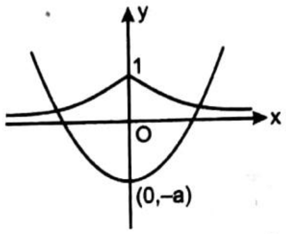
$\therefore \quad x = -1$, we get minimum value & $x = 1$, we get maximum value.

$$\left[-\frac{\pi}{4} - \frac{\pi}{4}, \frac{\pi}{4} + \frac{\pi}{4}\right] = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

4. $\left(2^{2x^2+2y} - 2^{2x+2y^2}\right)^2 = 1 - 2^{2x^2+2y^2+2x+2y+1} \ge 0$

$$\Rightarrow \quad \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 \le 0$$

5.



7. $\sec^{-1}\left(-\frac{5}{2}+\frac{2}{2(x^2+2)}\right)$ $\qquad x^2+2\geq 2$

$=\sec^{-1}\left(-\frac{5}{2}+\frac{1}{(x^2+2)}\right)$ $\qquad \left(\frac{1}{x^2+2}\leq\frac{1}{2}\right)$

$\leq \sec^{-1}(-2)=\pi-\sec^{-1}(2)$ $\qquad \left(-\frac{5}{2}+\frac{1}{x^2+2}\leq-\frac{5}{2}+\frac{1}{2}=-2\right)$

$=\frac{2\pi}{3}$

8. $f'(x)=x^2+ax+b$ is injective if $D\leq 0$

$a^2-4b\leq 0$

If	$a=1, b=1,2,3,4,5$	Number of pair $=5$
	$a=2, b=1,2,3,4,5$	Number of pair $=5$
	$a=3, b=3,4,5$	Number of pair $=3$
	$a=4, b=4,5$	Number of pair $=2$
	$a=5$	b has no value

9. $f(x)=\log_x[x] \Rightarrow f(x)\in[0,1]$

$g(x)=|\sin x|+|\cos x|$

$\Rightarrow\ g(x)\in[1,\sqrt{2}]$

10. $f(x)=2x^3-3x^2+6$

$f'(x)=6x^2-6x\geq 0. \qquad \Rightarrow x\in[1,\infty)$

and $f(x)\in[5,\infty)$

11. $0\leq\{x\}<1$

$\{x\}(\{x\}-1)(\{x\}+2)\geq 0$

$\Rightarrow\ \{x\}=0\ \Rightarrow x\in z$

14. $1+\sin^2 x\in[1,2]$

$\frac{1}{1+\sin^2 x}\in\left[\frac{1}{2},1\right]$

$\sin^{-1}\left(\frac{1}{1+\sin^2 x}\right)\in\left[\frac{\pi}{6},\frac{\pi}{2}\right]$

$\frac{K\pi}{6} \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right] \qquad K \in [1, 3]$

15. $f(x-y) = f(x)f(y) - f(a-x)f(a+y)$

Put $x = y = 0$

$f(0) = [f(0)]^2 - f(a)f(a)$

$\Rightarrow \quad f(a) = 0 \qquad [\because f(0) = 1]$

Put $x = a$ and $y = x$

$f(a-x) = f(a)f(x) - f(0)f(a+x)$

$\Rightarrow \quad -f(a-x) = f(a+x) \ \Rightarrow f(2a-x) = -f(x)$

18. $f(x) = 4x - x^2 = y$

$x^2 - 4x + y = 0$

$f^{-1}(x) = 2 - \sqrt{4-x}$

19. $[5\sin x] + [\cos x] = -6$

$\Rightarrow \quad -1 \le \cos x < 0$ and $-5 \le 5\sin x < -4$

$-1 \le \sin x < -\frac{4}{5}$

20. $f(x) = ax + \cos x$

$f'(x) = a - \sin x$

if $f(x)$ is invertible, then

$f'(x) \ge 0$ or $f'(x) \le 0$

$\Rightarrow \quad a \ge 1$ or $a \le -1$

21. $f(x) = [1 + \sin x] + \left[2 + \sin\frac{x}{2}\right] + \left[3 + \sin\frac{x}{3}\right] + \ldots + \left[n + \sin\frac{x}{n}\right]$

$= (1 + 2 + 3 + \ldots + n) + [\sin x] + \left[\sin\frac{x}{2}\right] + \left[\sin\frac{x}{3}\right] + \ldots + \left[\sin\frac{x}{n}\right]$

22. $y = \frac{x^2 + ax + 1}{x^2 + x + 1}$

$(y-1)x^2 + (y-a)x + (y-1) = 0$

$D \ge 0$

$(y-a)^2 - 4(y-1)^2 \ge 0$

$-3y^2 + y(8-2a) + a^2 - 4 \ge 0 \ \forall \ y \in R$

Not possible

23. $f(x) = [x] + [-x]$

$$f(x) = \begin{cases} 0 & x \in I \\ -1 & x \notin I \end{cases}$$

$g(x) = \{x\}$

$h(x) = f[g(x)] = f(\{x\}) \qquad \{x\} = 0 \quad x \in I$

$\qquad \{x\} = \{x\} \quad x \notin I$

$$h(x) = \begin{cases} f(0) & x \in I \\ f(\{x\}) & x \notin I \end{cases} \Rightarrow h(x) = \begin{cases} 0 & x \in I \\ -1 & x \notin I \end{cases}$$

Hence, the option (b).

24. $f(x) = \left[\frac{x}{15}\right]\left[-\frac{15}{x}\right] \qquad x \in (0, 90)$

$0 \le x < 15 \qquad f(x) = 0$

$15 \le x < 30 \qquad f(x) = -1$

$30 \le x < 45 \qquad f(x) = -2$

$45 \le x < 60 \qquad f(x) = -3$

$60 \le x < 75 \qquad f(x) = -4$

$75 \le x < 90 \qquad f(x) = -5$

Total integers in range $f(x) = \{0, -1, -2, -3, -4, -5\}$

25. $g(x) = \dfrac{1}{f(|x|)}$

$g(x) \Rightarrow$ even functions $\Rightarrow$ symmetric about y-axis

$\Rightarrow \quad x \to \infty \quad f(x) \to 0$

at $x = x_1 \qquad f(x) = 0 \Rightarrow g(x_1) \to \infty$

26. Homogeneous function $\Rightarrow f(tx, ty) = t^n f(x, y)$

27. $f(x) = \left[\begin{array}{ll} 2x+3 & x \le 1 \\ a^2x+1 & x > 1 \end{array}\right.$

For $x \le 1 \qquad f(x) \le 5$

So for range of $f(x)$ to be R.

$\Rightarrow \quad a^2 + 1 \le 5$ and $a \ne 0$

$\Rightarrow \quad a \in [-2, 2]$

Hence, $a = \{-2, -1, 1, 2\}$

28. $\log_{1/3}(\log_4(x-5)) > 0$

$0 < \log_4(x-5) < 1$

$1 < x - 5 < 4$

$6 < x < 9$

29. $f(x) = \log_2\left(\dfrac{4}{\sqrt{2+x}+\sqrt{2-x}}\right); -2 \le x \le 2$

$\sqrt{2+x}+\sqrt{2-x}=y$

$4+2\sqrt{4-x^2}=y^2$

$y\in[2,2\sqrt{2}]$

Range $f(x)=\left[\log_2\frac{4}{2\sqrt{2}},\log_2\frac{4}{2}\right]$

$f(x)$ lies between $\left[\frac{1}{2},1\right]$

30. $|x^2+5x|+|x-x^2|=|6x| \Rightarrow |x^2+5x|+|x-x^2|=|(x^2+5x)+(x-x^2)|$

$|a|+|b|=|a+b| \Rightarrow ab\geq 0$

$(x^2+5x)(x-x^2)\geq 0$

$x(x+5)\cdot x(x-1)\leq 0 \Rightarrow -5\leq x\leq 1$

31. $f(x)+f\left(\frac{1}{x}\right)=f(x)f\left(\frac{1}{x}\right)$

$\Rightarrow \quad f(x)=1\pm x^n$

$f(2)=33 \Rightarrow n=5$

Hence, $\quad f(x)=1+x^5$

Here, $\quad f(x)+f(-x)\neq 0.$

Hence not an odd function.

32. $g(x)=\frac{\sin x+\sin 7x}{\cos x+\cos 7x}+|\sin x|=\frac{2\sin 4x\cos 3x}{2\cos 4x\cos 3x}+|\sin x|$

$=\tan 4x+|\sin x|$

$g(x)$ period $=\pi$

33. $f(x)=\begin{cases}\frac{x-1}{2} & x=\text{odd}\\ -\frac{x}{2} & x=\text{even}\end{cases} \qquad f(x):N\to Z$

Let $x=\text{odd}=(2n+1); n>0$

$f(x)=\frac{2n+1-1}{2}=n \Rightarrow$ +ve integer

Let $x=\text{even}=2m; m>0$

$f(x)=-\frac{2m}{2}=-m \Rightarrow$ –ve integer

⇒ Range = codomains ⇒ onto and clearly $f(x)$ is one-one function.

Hence, bijective.

34. $y=\frac{2^{x+1}-2^{1-x}}{2^x+2^{-x}}=\frac{2^{2x+1}-2}{2^{2x}+1}$

$$y(2^{2x}+1)=(2^{2x}-1)2$$

$$2^{2x}\cdot y+y=2^{2x}\cdot 2-2$$

$$2^{2x}(y-2)=-(2+y)$$

$$2^{2x}=\frac{(y+2)}{(2-y)}$$

$$2x=\log_2\frac{(2+y)}{(2-y)}$$

$$x=\frac{1}{2}\log_2\frac{(2+y)}{(2-y)}$$

$$f^{-1}(x)=\frac{1}{2}\log_2\frac{(2+x)}{(2-x)}$$

35. $\sqrt{|y|}=x \qquad (\because x>0)$

$\Rightarrow \ |y|=x^2$

$\Rightarrow \ y=x^2 \qquad y\geq 0$

$y=-x^2 \qquad y<0$

36. $f(x)=\log_{[x]}(9-x^2)$

Domains $=\left.\begin{matrix}9-x^2>0\\ [x]>0 \text{ and } [x]\neq 1\end{matrix}\right\} \Rightarrow x\in[2,3) \Rightarrow f(x)=\log_2(9-x^2)$

Range $=(-\infty,\log_2 5]$

37. Gives $e^x+e^{f(x)}=e$

$e^{f(x)}=e-e^x$

$\Rightarrow \ f(x)=\log_e(e-e^x)$

Domain $e-e^x>0$

$x<1\Rightarrow x\in(-\infty,1)$

Range $(-\infty,1)$

38. Gives $y+|y|=x+|x|$

If $x>0, y>0 \Rightarrow 2y=2x \Rightarrow y=x$

$x<0, y>0 \Rightarrow 2y=0 \Rightarrow y=0$

$x>0, y<0 \Rightarrow 0=2x \Rightarrow x=0$

$x<0, y<0 \Rightarrow 0=0 \Rightarrow$ whole region of III quadrant.

For person to be safe there should not be point common to the given curves and the voltage field graph. Only $y=m+|x|$ does not have any point of intersection with the curve.

39. Gives $|f(x)+6-x^2|=|f(x)|+|4-x^2|+2$

$\Rightarrow \ |f(x)+2+(4-x^2)|=|f(x)|+|4-x^2|+2$

Since $|a+b+c| = |a| + |b| + |c|$.

If $a \geq 0, b \geq 0, c \geq 0$ or $a \leq 0, b \leq 0, c \leq 0$

$\Rightarrow$ $f(x) \geq 0$ and $4 - x^2 \geq 0 \Rightarrow -2 \leq x \leq 2$ and $f(x) \geq 0$

40. $f(x) = \cos px + \sin x$

Period : L.C.M. of $\left(\frac{2\pi}{p}, \frac{2\pi}{1}\right)$

For period to exist p should be a rational number.

41. $y = f(e^x) + f(\ln|x|)$

Domain $f(x) = (0, 1)$

$\Rightarrow$ $0 < e^x < 1 \quad \Rightarrow \quad x < 0$ ...(1)

and $0 < \ln|x| < 1 \quad \Rightarrow \quad 1 < |x| < e \Rightarrow x \in (-e, -1) \cup (1, e)$ ...(2)

Taking intersection $x \in (-e, -1)$

42. Givens $f(1) = 2,\ f(2) = 3,\ f(3) = 4,\ f(4) = 1,\ g(1) = 3$ and $f[g(x)] = g[f(x)]$

at $x = 1$, $f[g(1)] = g[f(1)] \quad \Rightarrow f(3) = g(2) \quad \Rightarrow g(2) = 4$

at $x = 2$ $f[g(2)] = g[f(2)] \quad \Rightarrow f(4) = g(3) \quad \Rightarrow g(3) = 1$

at $x = 3$ $f[g(3)] = g[f(3)] \quad \Rightarrow f(1) = g(4) \quad \Rightarrow g(4) = 2$

43. Gives $[y + [y]] = 2\cos x \quad \Rightarrow [y] + [y] = 2\cos x \Rightarrow 2[y] = 2\cos x;\ [y] = \cos x$ ...(1)

where
$$y = \frac{1}{3}[\sin x + [\sin x + [\sin x]]]$$
$$y = \frac{1}{3}[\sin x + [\sin x] + [\sin x]]$$
$$y = \frac{1}{3}(3[\sin x])$$
$$y = [\sin x] \qquad \text{...(2)}$$

From eqn. (1) & (2),
$$[\sin x] = \cos x$$
$\Rightarrow$ $\cos x = 0, 1, -1$

Hence, no solution.

44. $f(x) = \dfrac{x^{2n}}{(x^{2n}\,\mathrm{sgn}\,x)^{2n+1}}\left[\dfrac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}\right]$ $x \neq 0$ and $f(0) = 1$

when $f(x) = \dfrac{(x^{2n})}{(x^{2n})^{2n+1}}\left[\dfrac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}\right];\ x > 0$

$$f(x) = \frac{x^{2n}}{-(x^{2n})^{2n+1}}\left[\frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}\right]; \; x < 0$$

Clearly, $f(x) = f(-x)$. Hence, $f(x)$ is even function.

45. $f(n) = 2(f(1) + f(2) \ldots\ldots + f(n-1))$

$$f(2) = 2f(1)$$

$$f(3) = 2[f(1) + f(2)] = 2\left[\frac{f(2)}{2} + f(2)\right] = 3f(2)$$

$$f(4) = 2[f(1) + f(2) + f(3)] = 2\left[\frac{f(3)}{2} + f(3)\right] = 3f(3) = 3^2 f(2)$$

$$\vdots$$

$$\sum_{r=1}^{m} f(r) = f(1) + f(2) \ldots\ldots + f(m) = f(1) + f(2) + 3f(2) \ldots\ldots +$$

$$= f(1) + f(2)\,[1 + 3 + 3^2 \ldots\ldots 3^{m-2}]$$

$$= f(1) + 2 \cdot \frac{(3^{m-1} - 1)}{(3-1)} = 3^{m-1}$$

46. Gives

$$f(x) = \frac{x}{\sqrt{1+x^2}}$$

$$f(f(x)) = \frac{x}{\sqrt{1+2x^2}}$$

$$f(f(f(x))) = \frac{x}{\sqrt{1+3x^2}}$$

$$\vdots$$

$$\underbrace{fofo \ldots\ldots fof(x)}_{n \text{ times}} = \frac{x}{\sqrt{1+nx^2}} = \frac{x}{\sqrt{1 + \left(\sum_{r=1}^{n} 1\right)x^2}}$$

47. $f(x) = 2x + |\cos x|$

Range $f(x) = R$ = codomain ⇒ onto.

Clearly, $f(x)$ is increasing function ⇒ one-one function.

48. Gives $f(x) = x^3 + x^2 + 3x + \sin x$

Since, $f(x)$ is continuous function.

and $f(x) = \infty$ as $x \to \infty$

$f(x) = -\infty$ as $x \to -\infty$

Range $f(x) = R =$ codomains $\Rightarrow$ onto function

and $f'(x) = 3x^2 + 2x + 3 + \cos x = 3\left(x + \frac{1}{3}\right)^2 + \frac{8}{3} + \cos x \Rightarrow f'(x) > 0$

Hence, $f(x)$ is one-one.

49. $f(x) = \{x\} + \{x+1\} + \ldots \{x+99\}$

Since $\{x\} = \{x + I\}$ where $I =$ integer

$$f(x) = \underbrace{\{x\} + \{x\} \ldots \{x\}}_{100 \text{ times}}$$

$$f(x) = 100\{x\} \Rightarrow f(\sqrt{2}) = 100\{\sqrt{2}\} = 100 \times 0.414 = 41.4$$

$$[f(\sqrt{2})] = 41$$

50. $|\cot x + \operatorname{cosec} x| = |\cot x| + |\operatorname{cosec} x|$; $x \in [0, 2\pi]$

$\Rightarrow$ $\cot x \geq 0$ and $\operatorname{cosec} x \geq 0 \Rightarrow 1^{\text{st}}$ quadrant

or $\cot x \leq 0$ and $\operatorname{cosec} x \leq 0 \Rightarrow 4^{\text{th}}$ quadrant

Hence, $x \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right)$

51. If $f(4+x) = f(4-x)$

$\Rightarrow$ $f(x)$ is symmetric about $x = 4$.

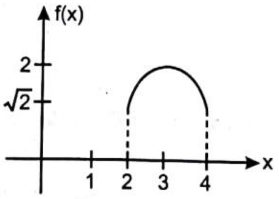
Roots of $f(x) = 0$ are of the form

$$4-\alpha, 4+\alpha, 4-\beta, 4+\beta, 4-\gamma, 4+\gamma, 4-\delta, 4+\delta$$

52. $f(x) + x - 6 = (x-1)(x-2)(x-3)(x-4)(x-5)$

$\Rightarrow$ $f(6) = 120$

53. $f(x) = \sqrt{x-2} + \sqrt{4-x}$



54. $\left[\frac{x}{9}\right] = \left[\frac{x}{11}\right]$

$x \in [1, 9) \cup [11, 18) \cup [22, 27) \cup [33, 36) \cup [44, 45)$

55. $\log_{\left[x+\frac{1}{2}\right]}(2x^2 + x - 1)$

$$\left[x + \frac{1}{2}\right] > 0, \left[x + \frac{1}{2}\right] \neq 1 \ \& \ 2x^2 + x - 1 > 0$$

$$\Rightarrow \quad x+\frac{1}{2}\geq 2 \;\&\; (2x-1)(x+1)>0$$

$$x\geq\frac{3}{2} \;\&\; x(-\infty,-1)\cup\left(\frac{1}{2},\infty\right)$$

$$\Rightarrow \quad x\in\left[\frac{3}{2},\infty\right)$$

56. $[x^2]+[x]-2=0$

Let $[x]=t$

$\Rightarrow \quad t^2+t-2=0$

$\Rightarrow \quad (t+2)(t-1)=0$

$\Rightarrow \quad t=-2$ or $t=1$

$\Rightarrow \quad [x]=-2$ or $[x]=1$

$\Rightarrow \quad x\in[-2,-1)\cup[1,2)$

58. $f(x)$ is many one function.

59. $f(f(x))=2+f(x) \qquad f(x)\geq 0$

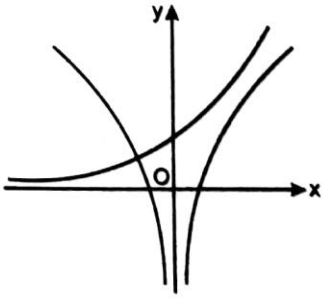
$\qquad =2-f(x) \qquad f(x)<0$

$f(f(x))=4+x \qquad x\geq 0$

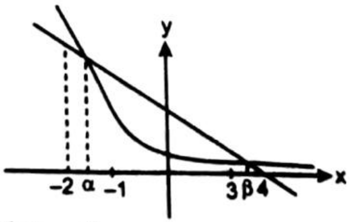
$\qquad =4-(x) \qquad x<0$

60. $f'(x)=\dfrac{7(3x^2-2x+3)}{(3+3x-4x^2)^2}>0\Rightarrow f(x)\uparrow$

61.



62.



$[\alpha]=-2$

$[\beta]=3$

63. $f(x)=\sqrt{\sin(\log_7(\cos(\sin x)))}$

$\cos(\sin x)\le 1 \Rightarrow \cos(\sin x)=1 \Rightarrow f(x)=0$

64. $-3\le|[x]|\le 2 \Rightarrow -2\le[x]\le 2 \Rightarrow -2\le x<3$

65. $f(x)=\frac{\pi}{2}+\cot^{-1}\{-x\}$

$0\le\{-x\}<1 \Rightarrow \frac{\pi}{4}<\cot^{-1}\{-x\}\le\frac{\pi}{2}$

66. $f(f(x))=x$

$f_{2008}(x)+f_{2009}(x)=x+f(x)=x+\frac{3x+5}{2x-3}=\frac{2x^2+5}{2x-3}$

67. $f(x)=\left(x+\frac{1}{x}+1\right)\left(x^2+\frac{1}{x^2}\right);\ x^2+\frac{1}{x^2}\ge 2;\ x+\frac{1}{x}+1\ge 3 \Rightarrow f(x)\ge 6$

68. $f(x)=e^{x^3-3x^2-9x+2}$

$f'(x)=e^{(x^3-3x^2-9x+2)}3(x-3)(x+1)$

$\Rightarrow$ $f(x)$ is many one.

at $x=-1$, $f(x)=e^7$

at $x\to-\infty$, $f(x)\to 0$

Range of $f(x)$ is $(0,e^7]$.

69. $D_f:(-2,1)$

$$-\infty<\log\left(\frac{\sqrt{4-x^2}}{1-x}\right)<\infty$$

$$-1\le\sin\left(\log\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right)\le 1$$

70. $f'(x)\ge 0\ \forall\ x\in R \Rightarrow 3x^2+2(a+2)x+3a\ge 0\ \forall\ x\in R$

$\Rightarrow D\le 0$

$\Rightarrow 4(a+2)^2-4\cdot 9a\le 0$

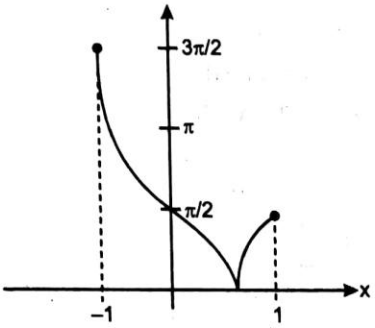
$\Rightarrow a^2-5a+4\le 0 \Rightarrow (a-1)(a-4)\le 0$

$\Rightarrow a\in[1,4]$

71. Min. value of $3x^2+bx+c=0$

$\Rightarrow D=0$

72. $f(x)=\sin^{-1}x-\cos^{-1}x=2\sin^{-1}x-\frac{\pi}{2}$

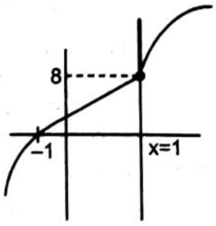


73. is one-one when

$$2^3 = \ln 1 + b^2 - 3b + 10$$

$$\Rightarrow \quad b^2 - 3b + 2 = 0$$

$$\Rightarrow \quad b = 1, 2$$



80. We have, $[x]^2 - 7[x] + 10 < 0$

$$\Rightarrow \quad ([x]-5)([x]-2) < 0$$

$$\Rightarrow \quad 2 < [x] < 5$$

$$\Rightarrow \quad [x] = 3 \text{ or } 4$$

$$\Rightarrow \quad x \in [3, 5)$$

and $\quad 4[y]^2 - 16[y] + 7 < 0$

$$(2[y]-7)(2[y]-1) < 0$$

$$\Rightarrow \quad \frac{1}{2} < [y] < \frac{7}{2}$$

$$\Rightarrow \quad [y] = 1 \text{ or } 2 \text{ or } 3$$

$$\Rightarrow \quad y \in [1, 4)$$

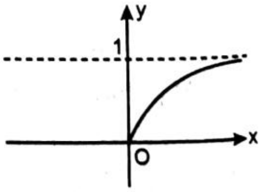
Therefore, $\quad x + y \in [4, 9)$

$$[x+y] \in \{4, 5, 6, 7, 8\}$$

Hence, $[x + y]$ cannot be 9.

81. $f: R \to R \qquad f(x) = \dfrac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$

$$f(x) = \begin{cases} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} & \text{if } x \ge 0 \\ \dfrac{e^{-x} - e^{-x}}{e^x + e^{-x}} & \text{if } x < 0 \end{cases}$$



Many one into function.

82. $f(x)$ such $f(1-x)+2f(x)=3x\ \forall\ x\in R$

$$x\to\left(\frac{1}{2}+x\right)$$

$$f\left(\frac{1}{2}-x\right)+2f\left(\frac{1}{2}+x\right)=3\left(\frac{1}{2}+x\right) \qquad \ldots(1)$$

$$x\to\left(\frac{1}{2}-x\right)$$

$$f\left(\frac{1}{2}+x\right)+2f\left(\frac{1}{2}-x\right)=3\left(\frac{1}{2}-x\right) \qquad \ldots(2)$$

(1) + (2)

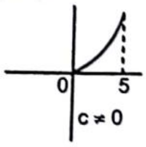
$$3\left(f\left(\frac{1}{2}+x\right)+f\left(\frac{1}{2}-x\right)\right)=3;\quad f\left(\frac{1}{2}-x\right)=1-f\left(\frac{1}{2}+x\right)$$

$$1+f\left(\frac{1}{2}+x\right)=\frac{3}{2}+3x;\quad f\left(\frac{1}{2}+x\right)=\frac{1}{2}+3x$$

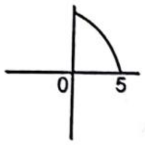
$$x=-\frac{1}{2}\Rightarrow f(0)=\frac{1}{2}-\frac{3}{2}=-1$$

83. $f:[0,5]\to[0,5]$

$f(x)=ax^2+bx+c$ $\qquad a,b,c\in R,\ abc\neq 0$



or



$$25a+5b+c=0$$

$$f(5)=0$$

$ax^2+bx+c=0\,(\alpha)$ $\qquad \dfrac{c}{a}=5\times\beta$

$cx^2+bx+a=0\left(\dfrac{1}{\alpha}\right)$ $\qquad \beta=\dfrac{1}{a}$

So, roots are $\left(a,\dfrac{1}{5}\right)$.

84. $f(x)=x^2+\lambda x+\mu\cos x$

$f(x)=x$

85. $f(k)=\text{odd}$

$f(k+1)=\text{even}\qquad k=1,2,3$

$$\left.\begin{aligned} f(1) &\Rightarrow \text{odd}\\ f(2) &\Rightarrow \text{even}\\ f(3) &\Rightarrow \text{odd}\\ f(4) &\Rightarrow \text{even}\end{aligned}\right\}\text{Hence, 4 functions.}$$

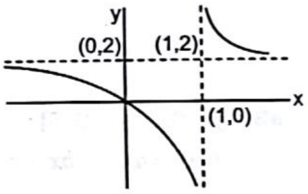
$$\left.\begin{aligned} f(1) &\Rightarrow \text{even}\\ f(2) &\Rightarrow \text{odd}\\ f(3) &\Rightarrow \text{even}\\ f(4) &\Rightarrow \text{odd}\end{aligned}\right\}\text{Hence, 4 functions.}$$

$$\left.\begin{aligned} f(1) &\Rightarrow \text{odd}\\ f(2) &\Rightarrow \text{even}\\ f(3) &\Rightarrow \text{even}\\ f(4) &\Rightarrow \text{odd}\end{aligned}\right\}\text{Hence, 4 functions.}$$

86. $y=\tan(\sin x)$.

Here function is continuous and differentiable and $y_{\max}=\tan(1)$; $y_{\min}=-\tan 1$.

87. $f(x)=\dfrac{2x}{x-1}$



$$y=2+\frac{2}{(x-1)}$$

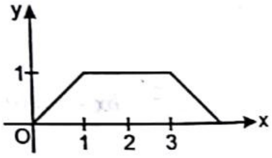
$$(y-2)(x-1)=2$$

88. $R_f=[-2,4]$

$R_g=[-1,2]$

89. $f(x)=(x^4+1)+\dfrac{1}{x^2+x+1}$

90. $0\le f(x)\le 1$



$0\le 7f(x)\le 7$

$-1\le \sin(7f(x))\le 1$

91. $\ln|\ln|x||\ge 0 \quad \cap \quad |x|^2-7|x|+10\le 0$

$|\ln|x||\ge 1 \qquad (|x|-2)(|x|-5)\le 0$

$\ln|x|\in(-\infty,-1)\cup[1,\infty) \qquad 2\le|x|\le 5$

$|x|\in\left(0,\dfrac{1}{e}\right]\cup[e,\infty) \qquad x\in(-5,-2]\cup[2,5]$

$$x\in(-\infty,-e]\cup\left[-\frac{1}{e},0\right)\cup\left(0,\frac{1}{e}\right]\cup[e,\infty)$$

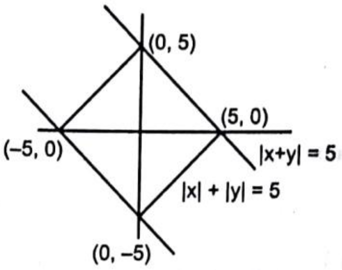
92. $\log_{[x]+3\{x\}}\left(\left([x]-\frac{5}{2}\right)^2+\frac{3}{4}\right)\geq 0 \Rightarrow [x]+3\{x\}>1$

93. $x-3=X$ $\qquad |X|+|Y|=5$

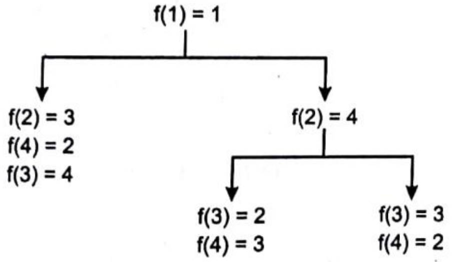
$y-1=Y$

$x+y-4=X+Y \qquad |X+Y|=5$

number of pairs of $(x, y)=12$



95.



96. $x^2-x\neq 0 \Rightarrow x\neq 0, 1$

97. Total one-one function – (at least one get right place) + (at least two get right place) – (at least three get right place) + (at least four get right place)

$$= {}^6C_4\times 4!-{}^4C_1\times {}^5C_3\times 3!+{}^4C_2\times {}^4C_2\times 2!-{}^4C_3\times {}^3C_1+{}^4C_4=181$$

98. $f(x)=x^2-2x-3$

$g(x)=f^{-1}(x)=1+\sqrt{x+4} \qquad x\geq -4$

$f(x)=g(x)=f^{-1}(x)\Rightarrow f(x)=x$

$\Rightarrow\ x^2-3x-3=0 \quad \Rightarrow x=\dfrac{3+\sqrt{21}}{2}$

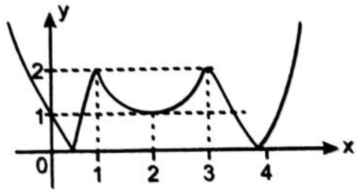
Exercise-2 : One or More than One Answer is/are Correct

1. $f(-4) = f(4) = 40$

$f(-13) = f(13) = f(3) = 19$

$f(-11) = f(11) = f(1) = 2$

2.



3. $f(x) = \cos^{-1}\left(\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\right)$ is defined when

$$\frac{x}{2} \neq (2n-1)\frac{\pi}{2}$$

$\Rightarrow$ $x \neq (2n-1)\pi$

Domain $= R - \{(2n-1)\pi : n \in I\}$

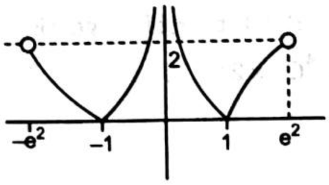
$\therefore$ Range $= [0, \pi)$

$f(x) = \cos^{-1}(\cos x)$

$f(x)$ is even function.

when $x \in (\pi, 2\pi)$, then $f(x) = 2\pi - x$ is differentiable.

4.



$0 < |k-1| - 3 < 2$

$\Rightarrow$ $k \in (-4, -2) \cup (4, 6)$

5. (a) $D_f \in R$

(b) $D_f \in R$

(c) $f(x)=\sqrt{2\cos^2 x+\cos x+\frac{1}{8}}$

$D_f \in R$

(d) $\ln(1+|x|)\geq 0$

$D_f \in \left\{\frac{(2n+1)\pi}{2}\right\}$

6. $f\left(\frac{3}{2}\right)=\frac{9}{4}$

$f\left(f\left(\frac{3}{2}\right)\right)=\frac{3}{2}$

$f\left(f\left(f\left(\frac{3}{2}\right)\right)\right]=\frac{9}{4}$

$f\left(\frac{5}{2}\right)=2$

$f\left(f\left(\frac{5}{2}\right)\right)=1$

$f\left(f\left(f\left(\frac{5}{2}\right)\right)\right)=1$

8. $f^{-1}(f(x))=f(f^{-1}(x))=x$

if $f(f^{-1}(x))=f^{-1}(x) \Rightarrow x=f^{-1}(x)$

if $f(f^{-1}(x))=f^{-1}(x) \Rightarrow f(f^{-1}(f(x)))=f^{-1}(f(x)) \Rightarrow f(x)=f^{-1}(f(x))=x$

9. $f(x)=\cos^{-1}x+\cos^{-1}\left(\frac{x}{2}+\frac{\sqrt{3}\cdot\sqrt{1-x^2}}{2}\right)$

Let $x=\cos\theta$

$$f(x)=\cos^{-1}(\cos\theta)+\cos^{-1}\left(\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right)$$

$$=\cos^{-1}(\cos\theta)+\cos^{-1}\left(\cos\left(\theta-\frac{\pi}{3}\right)\right)$$

$$=\frac{\pi}{3} \qquad 0\leq\theta\leq\frac{\pi}{3}$$

$$=2\theta-\frac{\pi}{3} \qquad \frac{\pi}{3}<\theta\leq\pi$$

10. $f(x) = \cos^{-1}(-\{-x\})$

$-\{-x\} \in (-1, 0] \Rightarrow \cos^{-1}(-\{-x\}) \in \left[\frac{\pi}{2}, \pi\right)$

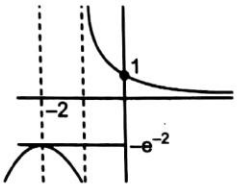
12. $h(x) = [\ln x - 1] + [1 - \ln x]$

$\Rightarrow h(x) = \begin{cases} -1, & \ln x - 1 \notin I \\ 0, & \ln x - 1 \in I \end{cases}$

14. $f(x) = \frac{1}{2}$, $f(x)$ is periodic & constant function.

16. $f(x) = \frac{e^{-x}}{1+x}$

$f'(x) = \frac{-e^{-x}(x+2)}{(1+x)^2}$



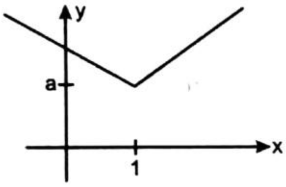
17. $[x] = \frac{2x\{x\}}{x+\{x\}} = \frac{2\{x\}([x]+\{x\})}{[x]+2\{x\}}$

$\Rightarrow \quad [x]^2 = 2\{x\}^2$

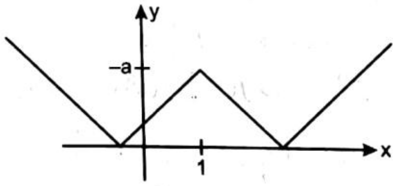
$\Rightarrow \quad x = 1 + \frac{1}{\sqrt{2}} \qquad (\because x \in R^+)$

18. $||x-1| + a| = 4$

if $a > 0$



if $a < 0$



(a) if eq. has three distinct real root then $a < 0$ and $a = -4$

(b) 4 distinct roots for $a \in (-\infty, -4)$

(c) if $-4 < a < 4$, there are two distinct real roots

(d) if $a > 4$, no real root.

19. (a) $f_2(x) = (\sin x)^{1/2} + (\cos x)^{1/2}$

$\sqrt{\sin x} > \sin^2 x; \quad \sqrt{\cos x} > \cos^2 x \quad \Rightarrow \sqrt{\sin x} + \sqrt{\cos x} > 1$

(b) $f_2(x) = (\sin x)^{1/2} + (\cos x)^{1/2} \qquad \Rightarrow f_2(x) = 1$ at $x = 2k\pi$

(c) $f_2(x) = (\sin x)^{1/2} + (\cos x)^{1/2}$; $f_3(x) = (\sin x)^{1/3} + (\cos x)^{1/3}$

if $x \in (2k\pi, 2k\pi + \pi/2) \qquad 0 < \sin x < 1$ and $0 < \cos x < 1$

As power increases, value of function decreases.

$$\Rightarrow \quad f_2(x) < f_3(x)$$

(d) $f_3(x) = (\sin x)^{1/3} + (\cos x)^{1/3}$

$$f_5(x) = (\sin x)^{1/5} + (\cos x)^{1/5}$$

$$\Rightarrow \quad f_3(x) < f_5(x)$$

20. $-1 \le \log_3\left(\dfrac{x^2}{3}\right) \le 1 \;\Rightarrow\; \dfrac{1}{3} \le \dfrac{x^2}{3} \le 3$

Range is $[0, 1]$.

21. $\dfrac{3x-1}{2} = n$

$$\left[\frac{4n+5}{9}\right] + \left[\frac{4n+5}{9} + \frac{1}{2}\right] = n$$

$$\left.\begin{array}{l} n = 2 \\ \;\vdots \\ \;\vdots \\ n = 10 \end{array}\right\}$$

22. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

$$= 1 - \frac{3}{4}\left(\frac{1-\cos 4\theta}{2}\right) = \frac{5}{8} + \frac{3}{8}\cos(4\theta)$$

$$= \frac{5}{8} + \frac{3}{8}\cos(x)$$

23. (a) $g(f(x)) = \ln(\sin x)$

(b) $x^2 + (a-1)x + 9 > 0 \;\forall\; x \in R$

$$(a-1)^2 - 36 < 0 \Rightarrow -5 < a < 7$$

(c) $f(f(x)) = (2011 - (2011 - x^{2012}))^{1/2012} = x$

24. $\left[\dfrac{1}{4} + \dfrac{150}{200}\right] + \left[\dfrac{1}{4} + \dfrac{151}{200}\right] + \ldots + \left[\dfrac{1}{4} + \dfrac{199}{200}\right] = 50$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 4 to 6

Sol. $f(x) = \sqrt{\theta x^2 - 2(\theta^2 - 3)x - 12\theta}$

$g(x) = \ln(x^2 - 49)$

if domain of $f + g$ is same as domain of g. Then

$$\theta x^2 - 2(\theta^2 - 3)x - 12\theta \ge 0 \;\forall\; x \in (-\infty, -7) \cup (7, \infty)$$

$$\Rightarrow \qquad \theta \in \left[\frac{6}{7}, \frac{7}{2}\right]$$

$$h(\theta) = \ln\left[\int_0^{\theta} 4\cos^2 t \, dt - \theta^2\right] = \ln[2\theta + \sin 2\theta - \theta^2]$$

Paragraph for Question Nos. 7 to 8

7. For $x \in [5^4, 5^5]$

$$f(x) = \alpha^4\left[2 - \left|\frac{x}{5^4} - 3\right|\right]$$

$\alpha = 2$

$f(x)_{\max} = 32$

8. $\alpha = 5$

$$f(x) = 5^4\left[2 - \left|\frac{x}{5^4} - 3\right|\right]$$

$$f(2007) = 5^4\left[2 - \frac{2007}{625} + 3\right] = 1118$$

Paragraph for Question Nos. 9 to 10

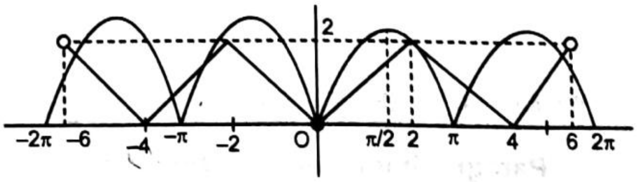
9. $$f(x) \begin{cases} \to x & 0 \le x \le 2 \\ \to -x & -2 \le x < 0 \end{cases}$$

$f(x) = f(x+4)$

$\{f(5.12)\} = \{f(1.12)\} = 0.12$

$\{f(7.88)\} = \{f(3.88)\} = \{f(-0.12)\} = 0.12$

10.



Paragraph for Question Nos. 13 to 14

13. $f(x) = 3$

$3 + \ln b_1, 3 + \ln b_2, 3 + \ln b_3$ are in A.P.

14. $y = 3x^2$

Let slope of tangent be m.

$$\Rightarrow \quad y = m(x-2)$$
$$\Rightarrow \quad m(x_1 - 2) = 3x_1^2$$
$$\text{Also,} \quad m = 6x_1$$
$$\Rightarrow \quad 6x_1(x_1 - 2) = 3x_1^2$$
$$x_1 = 4$$
$$m = 24$$

Paragraph for Question Nos. 15 to 16

15. $y = 2^{x^4 - 4x^2} \Rightarrow x^4 - 4x^2 = \log_2 y$

$$x^2 = \frac{4 + \sqrt{16 - 4\log_2 y}}{2} \Rightarrow x = \sqrt{2 + \sqrt{4 - \log_2 y}}$$

16. $g(x) = 1 + \dfrac{6}{\sin x - 2} \Rightarrow$ Range $[-5, -2]$

Exercise-4 : Matching Type Problems

1.
$$[x] + \{x\} + [y] + \{z\} = 12.7 \quad \ldots\text{(i)}$$
$$[x] + \{y\} + [z] + \{z\} = 4.1 \quad \ldots\text{(ii)}$$
$$\{x\} + [y] + \{y\} + [z] = 2 \quad \ldots\text{(iii)}$$

Adding (i), (ii) & (iii),

$$\Rightarrow \quad [x] + \{x\} + [y] + \{y\} + [z] + \{z\} = 9.4$$
$$\Rightarrow \quad \{y\} + [z] = -3.3,\ \{x\} + [y] = 5.3,\ [x] + \{z\} = 7.4$$
$$\Rightarrow \quad \{y\} = 0.7,\ [z] = -4,\ \{x\} = 0.3,\ [y] = 5$$
$$[x] = 7,\ \{z\} = 0.4$$

4. (A) $f(x) = \sin^2 2x - 2\sin^2 x = 2\sin^2 x \cos 2x$

Function is even, hence many one, function is also periodic.

$$f(x) = (1 - \cos 2x)\cos 2x = \frac{1}{4} - \left(\cos 2x - \frac{1}{2}\right)^2$$

Range of function is $\left[-2, \dfrac{1}{4}\right]$.

(B) $f(x) = 4x$

(C) $f(x) = \sqrt{\ln(\cos(\sin x))}$

$$\ln(\cos(\sin x)) \geq 0$$
$$\Rightarrow \quad \cos(\sin x) = 1$$
$$\Rightarrow \quad f(x) = 0$$

(D) $f(x) = \tan^{-1}\left(\dfrac{x^2+1}{x^2+\sqrt{3}}\right)$

$f(x)$ is even & hence many one.

Range is $\left[\dfrac{\pi}{6}, \dfrac{\pi}{4}\right)$.

7. (A) Domain of $g(x)$ is $[0, 3]$.

(B) Range of $g(x)$ is $[0, 3]$.

(C) $f(f(f(2))) = 1$

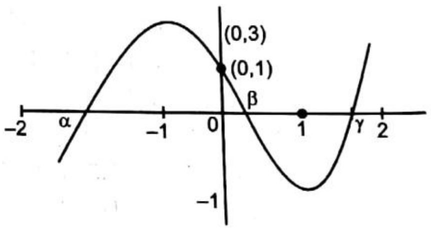
$f(f(f(3))) = 2$

(D) $m = 3$

Exercise-5 : Subjective Type Problems

1. $f(x) - 2x + 1 = (x-1)(x-2)(x-3)(x-4)(x-5)(2009x - \alpha)$

2. $f(x) = x^3 - 3x + 1$



$f(f(x)) = 0$

Let $f(x) = t$

$\Rightarrow \quad f(t) = 0$

$\Rightarrow \quad t = \alpha, \beta, \gamma$

$\Rightarrow \quad f(x) = \alpha,\ \alpha \in (-2, -1)$

No. of solution $= 1$

$f(x) = \beta,\ \beta \in (0, 1)$

No. of solution $= 3$

$f(x) = \gamma,\ \gamma \in (1, 2)$

No. of solution $= 3$

3. Put $x = y = 0$ $\quad f(1) = 4$

Put $x = 0, y = 1$ $\quad f(2) = 9$

4. $-1 \le \dfrac{2x}{3} \le 1 \quad \Rightarrow \dfrac{-3}{2} \le x \le \dfrac{3}{2}$

$12 - 3^x - \dfrac{27}{3^x} \ge 0 \Rightarrow (3^x - 3)(3^x - 9) \le 0 \Rightarrow 1 \le x \le 2$

5. $\sin^{-1}(0) + \cos^{-1}(-1) = \pi \qquad 0 \le x^2 < \dfrac{4}{9}$

$\sin^{-1}(1) + \cos^{-1}(0) = \pi \qquad \dfrac{4}{9} \le x^2 < \dfrac{13}{9}$

8. Let $P(x) = ax^4 + bx^3 + cx^2 + dx + 2$

$$P(1) = a + b + c + d + 2 = 5 \quad \ldots(1)$$

$$P(-1) = a - b + c - d + 2 = 5 \quad \ldots(2)$$

$$\Rightarrow \quad b + d = 0 \text{ and } a + c = 3$$

$$P(2) = 16a + 8b + 4c + 2d + 2 = 2 \quad \ldots(3)$$

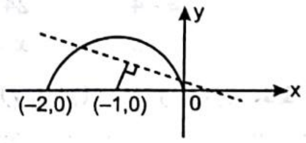
$$P(-2) = 16a - 8b + 4c - 2d + 2 = 2 \quad \ldots(4)$$

$$\Rightarrow \quad 4a + c = 0 \quad \text{and} \quad 4b + d = 0$$

$$\Rightarrow \quad b = d = 0 \quad \text{and} \quad a = -1, c = 4$$

$$\Rightarrow \quad P(x) = -x^4 + 4x^2 + 2$$

9. $(x+1)^2 + y^2 = 1 \qquad (\because y > 0)$



$$x + y = k$$

$$\left|\frac{k+1}{\sqrt{2}}\right| < 1$$

$$-\sqrt{2} - 1 < k < \sqrt{2} - 1$$

$$\Rightarrow \quad 0 < k < \sqrt{2} - 1 \qquad (\because k > 0)$$

10. $\sqrt{[x] + \left[\frac{x}{2}\right]} + \left[\frac{x}{3}\right] = 3$

$\therefore$ when $\left[\frac{x}{3}\right]$ is an integer then definitely $\sqrt{[x] + \left[\frac{x}{2}\right]}$ is also an integer.

So, $\sqrt{[x] + \left[\frac{x}{2}\right]} = 2$ and $\left[\frac{x}{3}\right] = 1$ (and check like this)

$$[x] + \left[\frac{x}{2}\right] = 4, \quad \left[\frac{x}{3}\right] = 1 \Rightarrow x \in [3, 6)$$

when $x \in [3, 4)$

$[x] = 3, \left[\frac{x}{2}\right] = 1$

So, $x \in [3, 4)$ satisfies.

when $x \in [4, 5)$ $[x] = 4 \left[\frac{x}{2}\right] = 2 \Rightarrow [x] + \left[\frac{x}{2}\right] = 6 \neq 4$ not satisfies, similarly on checking all possibilities we have only $x \in [3, 4)$.

$\therefore \quad a = 3, b = 4$

11. $f(f(x)) = \dfrac{1}{\sqrt[2011]{1 - \dfrac{1}{1 - x^{2011}}}} = \dfrac{\sqrt[2011]{1 - x^{2011}}}{-x}$

$$f(f(f(x))) = \frac{\sqrt[2011]{\left(1-\frac{-1}{1-x^{2011}}\right)}}{\frac{-1}{\sqrt[2011]{1-x^{2011}}}} = \frac{\frac{-x}{\sqrt[2011]{1-x^{2011}}}}{\frac{-1}{\sqrt[2011]{1-x^{2011}}}} = x$$

$\therefore \quad f_{2013}(x) = x = \{-x\}$

12.
$$\begin{array}{lll} f(x) & = 0 & 0 < x < 6 \\ & = -1 & 6 \le x < 12 \\ & = -2 & 12 \le x < 18 \\ & = -3 & 18 \le x < 24 \\ & = -4 & 24 \le x < 30 \\ & = -5 & x = 30 \end{array}$$

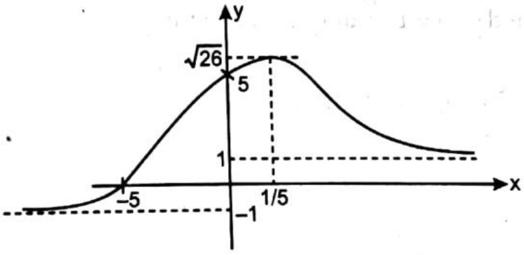
13. $(f(x,y))^2 - (g(x,y))^2 = \frac{1}{2}$

$f(x,y) \cdot g(x,y) = \frac{\sqrt{3}}{4}$

$\Rightarrow \quad f(x,y) = x^2 - y^2 = \pm\frac{\sqrt{3}}{2}$

$g(x,y) = 2xy = \pm\frac{1}{2}$

14. $f(x) = \frac{x+5}{\sqrt{x^2+1}}$



15. $f(x)$ is injective for $x \in \left(-\infty, \frac{1}{5}\right]$

$[\lambda] = \left[\frac{1}{5}\right] = 0$

16. $f: R \to R \; f(x) = \frac{x^3}{3} + (m-1)x^2 + (m+5)x + n$

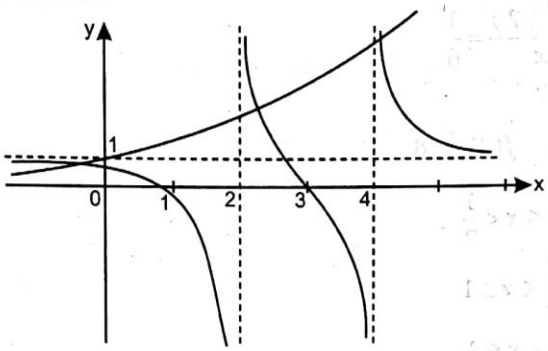
$f'(x) = x^2 + 2(m-1)x + (m+5) \ge 0$

$\Delta \le 0$

$$4(m-1)^2-4(m+5)\le 0$$
$$m^2-3m-4\le 0$$
$$(m-4)(m+1)\le 0$$
$$-1\le m\le 4$$

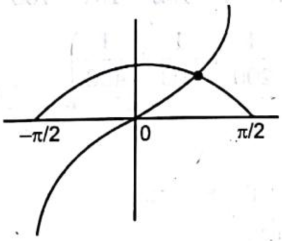
17. $f(x)=\dfrac{(x-1)(x-3)}{(x-2)(x-4)}-e^x$

$f(x)=0$ has three solutions.



$f(-x)=\dfrac{(x+1)(x+3)}{(x+2)(x+4)}-e^{-x}=0$ has three solutions.

$x^3=\cos x$

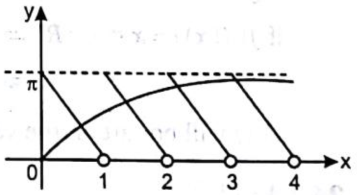


one solution

there are total 7 solutions.

18: $\cos^{-1}\left(\dfrac{2}{(1+x)^2}-1\right)=\pi(1-\{x\})$

there are total 76 solutions.



19. $f(x)=x^2-bx+c=0 \begin{matrix}\nearrow p_1\\ \searrow p_2\end{matrix}$

$p_1+p_2=b$ (odd no.)

$\Rightarrow\quad p_1=2$

$$p_1p_2 = c$$

$$b + c = (p_2 + 2) + 2p_2 = 35$$

$$\Rightarrow \quad p_2 = 11$$

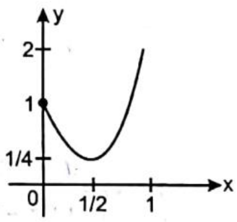
$$\Rightarrow \quad f(x) = x^2 - 13x + 22$$

$$\lambda = f(x)_{\min} = -\frac{81}{4}$$

20. $$f'(x) = \lim_{x\to 0} \frac{f(x) - f\left(\frac{x}{7}\right)}{x - \frac{x}{7}} = \frac{1}{6}$$

$$\Rightarrow \quad f(x) = \frac{x}{6} + 1 \;\Rightarrow\; f(42) = 8$$

21. $$g(x) = \begin{cases} f(x) & 0 \le x < \frac{1}{2} \\ = \frac{1}{4} & \frac{1}{2} \le x \le 1 \\ = 3 - x & 1 < x \le 2 \end{cases}$$



22. $$x = \frac{10}{4}\sum_{r=3}^{100}\left(\frac{1}{r-2} - \frac{1}{r+2}\right) = \frac{10}{4}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{102} - \frac{1}{101} - \frac{1}{100} - \frac{1}{99}\right)$$

$$= 5 \times 49\left(\frac{1}{99} + \frac{1}{200} + \frac{1}{303} + \frac{1}{408}\right)$$

$$[x] = 5$$

23. $f(x) = x$ has two real roots.

$$cx^2 + (d - a)x - b = 0 \begin{matrix} \nearrow 7 \\ \searrow 11 \end{matrix}$$

$$\frac{a-d}{c} = 18 \text{ and } \frac{-b}{c} = 77$$

if $f(f(x)) = x \;\forall\; x \in R \;\Rightarrow (ac + cd)x^2 + (d^2 - a^2)x - (a + d)b = 0$

$$\Rightarrow a + d = 0 \;\Rightarrow\; a = -d$$

$f(x)$ will not attain the value $\frac{a}{c} = 9$.

24. $A = (1, 3)$

$p \le -2^{1-1},\; p \le -2^{1-3}$

$1 - 2(p + 7) + 5 \le 0$ and $9 - 6(p + 7) + 5 \le 0 \Rightarrow p \in [-4, -1]$

25. $y=\dfrac{x-\dfrac{1}{x}}{x^3-\dfrac{1}{x^3}+2}$ Let $t=x-\dfrac{1}{x}>0$ for $x>1$

$y=\dfrac{t}{t(t^2+3)+2}$ $x^3-\dfrac{1}{x^3}=t(t^2+3)$

$=\dfrac{t}{t^3+3t+2}$

$=\dfrac{1}{t^2+\dfrac{2}{t}+3}$ $\left(\begin{array}{l} t^2+\dfrac{2}{t}=t^2+\dfrac{1}{t}+\dfrac{1}{t}\ge 3 \\ \therefore t^2+\dfrac{2}{t}+3\ge 6 \text{ (AM} \ge \text{GM)} \end{array}\right.$

$y_{\max}=\dfrac{1}{\left(t^2+\dfrac{2}{t}+3\right)_{\min}}=\dfrac{1}{6}$

$p=1, q=6$

28. $a+ar+ar^2=1$

$a^2r+a^2r^2+a^2r^3=\beta=ar(a+ar+ar^2)=ar$

$a^3r^3=-\gamma$

29. $m={}^6C_4\times 1=15$

$n=\dfrac{6!}{3!1!1!1!3!}\times 4!+\dfrac{6!}{(2!)^4}\times 4!=1560$

30. $\sum_{r=1}^{n}[\log_2 r]=0+1+1+(2+2+2+2)+\underbrace{(3+3+\ldots+3)}_{8\text{ times}}+\ldots$

$=2\cdot 1+4\cdot 2+8\cdot 3+\ldots+$

32. 3

$|(x-2y)(y+x)(x+3y)|=f(x,y)$

No rain, then $f(x,y)=0$ hence 3 lines.

33. Cubic $=(x^2-5x+6)(x+\alpha)+2(Bx+100-4\alpha)$

$(x^2-5x+4)(x+\alpha)+Bx+100-4\alpha$

Both identical $B=-2$

$\alpha=50$

Cubic $=(x^2-5x+6)(x+50)-4x-200$

34. $f(\theta) = 0 \Rightarrow \theta = -5 \pm \sqrt{5}$

$\Rightarrow f(f(f(x))) = -5 \pm \sqrt{5}$

Since $f(x) = (x+5)^2 - 5$

$f(f(f(x))) = -5 \pm \sqrt{5}$

$((f(f(f(x)))) + 5)^2 = -5 \pm \sqrt{5}$

$(f(f) + 5)^2 = \sqrt{5}$

$f(f) + 5 = \pm 5^{1/4}$

$f(f) = -5 \pm 5^{1/4}$

$(f+5)^2 - 5 = -5 \pm 5^{1/4}$

$(f+5)^2 = 5^{1/4}$

$f + 5 = \pm 5^{1/8}$

35. Let $\ln x = t$

$$y = \frac{2t^2 + 3t + 3}{t^2 + 2t + 2} \Rightarrow (y-2)t^2 + (2y-3)t + (2y-3) > 0$$

$$D \geq 0 \Rightarrow (2y-3)(2y-5) \leq 0 \Rightarrow \frac{3}{2} \leq y < \frac{5}{2}$$

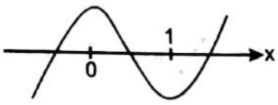
36. $P(x) = (x-3)Q_1(x) + 6 \Rightarrow P(3) = 6$ ...(1)

$P(x) = (x^2 - 9)Q(x) + (ax + b)$

$P(3) = 3a + b = 6$

If equation of odd degree polynomial, then $b = 0, a = 2$.

37. $f(x) = 2x^3 - 3x^2 + P$



$f'(x) = 6x^2 - 6x = 6x(x-1)$

$f(0) \geq 0 \cap f(1) \leq 0$

$\Rightarrow P \geq 0 \cap P - 1 \leq 0$

38. $f(x) = \dfrac{1}{\sqrt{\ln(\cos^{-1} x)}}$

$\ln(\cos^{-1} x) > 0 \Rightarrow \cos^{-1} x > 1$

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Chapter 2 – Limit

2

LIMIT

Exercise-1 : Single Choice Problems

1. $$\lim_{x\to0}\frac{2\sin\left(\frac{x-\tan x}{2}\right)\sin\left(\frac{\tan x+x}{2}\right)}{\left(\frac{x-\tan x}{2}\right)\left(\frac{\tan x+x}{2}\right)}\times\left(\frac{x-\tan x}{x^3}\right)\left(\frac{x+\tan x}{x}\right)\times\frac{1}{4}$$

$$=\frac{1}{2}\times\left(-\frac{1}{3}\right)\cdot2=-\frac{1}{3}$$ (use expansions)

3. $$a=\lim_{x\to0}\left(\frac{\ln(1+\cos2x-1)}{\cos2x-1}\right)\frac{(\cos2x-1)}{3x^2}=-\frac{2}{3}$$

$$b=\lim_{x\to0}\left(\frac{\sin^2 2x}{4x^2}\right)\frac{4x^2}{x^2\left(\frac{1-e^x}{x}\right)}=-4$$

$$c=\lim_{x\to1}\frac{\sqrt{x}(1-x)}{\left(\frac{\ln(1+x-1)}{x-1}\right)(x-1)(\sqrt{x}+1)}=\frac{-1}{2}$$

4. $f(x)=\frac{\pi}{2}-3\tan^{-1}x$

$g(x)=2\tan^{-1}x$

$$\lim_{x\to0}\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(a)}{g'(a)}=-\frac{3}{2}$$

5. $$\lim_{x\to0}\left(e^{\frac{2}{x}\ln(1+x)-2}\right)^{\frac{4}{\sin x}}=e^{\lim\limits_{x\to0}\frac{4}{\sin x}\left(e^{2\left(\frac{\ln(1+x)}{x}-1\right)}-1\right)}$$

$$=e^{\lim\limits_{x\to0}\frac{4}{\sin x}\left(\frac{e^{2\left[\frac{\ln(1+x)}{x}-1\right]}-1}{2\left(\frac{\ln(1+x)}{x}-1\right)}\right)\times2\left[\frac{\ln(1+x)}{x}-1\right]}$$

$$=e^{\lim\limits_{x\to0}8\left(\frac{x-\frac{x^2}{2}+\dots}{x}-1\right)\times\frac{1}{\sin x}}$$

$$=e^{-\frac{8}{2}}=e^{-4}$$

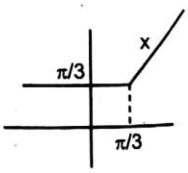
6. $$\lim_{x\to\infty}\frac{3}{x}\left(\frac{x}{4}-\left\{\frac{x}{4}\right\}\right)=\frac{3}{4}-0=\frac{3}{4}$$

$\Rightarrow \qquad p+q=7$

7. $$f(x)=\lim_{n\to\infty}\frac{x\left(1+\left(\frac{\pi}{3x}\right)^n\right)}{1+\left(\frac{\pi}{3x}\right)^{n-1}}=x;\ x>\frac{\pi}{3}$$

$$=\lim_{n\to\infty}\frac{\frac{\pi}{3}\left(\left(\frac{3x}{\pi}\right)^n+1\right)}{\left(\left(\frac{3x}{\pi}\right)^{n-1}+1\right)}=\frac{\pi}{3};\ x<\frac{\pi}{3}$$

$$=\frac{\pi}{3} \qquad x=\frac{\pi}{3}$$



$\therefore \quad f(x)=x;\ x\ge\frac{\pi}{3}$

$\qquad =\frac{\pi}{3};\ x<\frac{\pi}{3}$

$\therefore$ Option (d) is wrong.

8. $$\lim_{x\to0}\frac{\sin(\pi-\pi\cos^2(\tan(\sin x)))}{x^2}=\lim_{x\to0}\frac{\sin[\pi\sin^2(\tan(\sin x))]}{\pi\sin^2(\tan(\sin x))}\times\pi\left(\frac{\sin(\tan(\sin x))}{x}\right)^2=\pi$$

9. $$\lim_{x\to3^-}f(x)=\lim_{x\to3^+}f(x)\Rightarrow\lim_{x\to3^-}\frac{(27)^{\frac{(x+3)x}{27}}-9}{3^x-27}=\lim_{x\to3^+}\lambda\frac{1-\cos(x-3)}{(x-3)^2}$$

$$\Rightarrow \quad \lim_{x\to 3^-} \frac{3^2\left(3^{\frac{x^2+3x}{9}-2}-1\right)}{3^3(3^{x-3}-1)} = \frac{\lambda}{2}$$

$$\Rightarrow \quad \lim_{x\to 3} \frac{1}{3}\,\frac{x^2+3x-18}{9(x-3)} = \frac{\lambda}{2} \Rightarrow \frac{1}{27}\cdot 9 = \frac{\lambda}{2} \Rightarrow \lambda = \frac{2}{3}$$

10. $$\lim_{x\to\frac{\pi}{3}} \frac{2\sin\left(\frac{\frac{\pi}{3}-x}{2}\right)\cos\left(\frac{\frac{\pi}{3}-x}{2}\right)}{2\left(\cos x-\cos\frac{\pi}{3}\right)} = \lim_{x\to\frac{\pi}{3}} \frac{\sin\left(\frac{\frac{\pi}{3}-x}{2}\right)\cos\left(\frac{\frac{\pi}{3}-x}{2}\right)}{2\sin\left(\frac{\frac{\pi}{3}-x}{2}\right)\sin\left(\frac{\frac{\pi}{3}+x}{2}\right)}$$

$$= \frac{1}{2}\,\frac{1}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

11. $$\lim_{x\to\frac{\pi}{2}} \frac{\sin x}{\cos^{-1}[\sin^3 x]} \Rightarrow \frac{\sin\frac{\pi}{2}}{\cos^{-1}(0)} = \frac{1}{\pi/2} = \frac{2}{\pi}$$

13. $$\lim_{x\to I^-}\{x\} = \lim_{x\to I^-} x-[x] = 1; \quad \lim_{x\to I^-} \frac{e^{\{x\}}-\{x\}-1}{\{x\}^2} = e-2$$

16. $$\lim_{x\to\infty} x\left[x^{5c-1}\left(1+\frac{7}{x}+\frac{2}{x^5}\right)^c-1\right] = l$$

Case-I: $5c-1>0$, then $l\to\infty$

Case-II: $5c-1<0$, then $l\to-\infty$

Since limit is finite and non-zero so $5c-1=0 \Rightarrow c=\frac{1}{5}$

$$\therefore \quad \lambda = \lim_{x\to\infty} x\left[\left(1+\frac{7}{x}+\frac{2}{x^5}\right)^{1/5}-1\right]$$

$$= \lim_{x\to\infty} x\left[1+\left(\frac{1}{5}\right)\left(\frac{7}{x}+\frac{2}{x^5}\right)+\ldots\ldots-1\right]$$ (by binomial approximation)

$$= \frac{7}{5}$$

17. $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x^2}\left(\dfrac{\cos x - 1}{x^{n-2}} - \dfrac{(e^x - 1)}{x^{n-2}}\right) = 0 \Rightarrow n = 1, 2, 3$

18. 1^∞ (form) $= e^{\lim\limits_{x\to 0}\frac{1}{1-\cos x}\left(\frac{\sin x - x}{x}\right)} = e^{2\times -1/6} = e^{-1/3}$

19. $\lim\limits_{x\to\infty} [\sqrt{x^2 - x + 1} - (ax + b)] = 0$

So $a > 0$, on rationalizing

$$\lim_{x\to\infty}\left[\frac{(x^2 - x + 1) - [a^2x^2 + b^2 + (2ab)x]}{\sqrt{x^2 - x + 1} + ax + b}\right] = 0$$

So, $1 - a^2 = 0 \qquad -1 - 2ab = 0$

$a = 1$

$\lim\limits_{n\to\infty} \sec^2[k!\pi(-1/2)] = 1 = a$

20. $f(x + T) = f(x + 2T) = \ldots\ldots = f(x + nT) = f(x)$

$$\lim_{n\to\infty} \frac{nf(x)(1 + 2 + 3 + \ldots + n)}{f(x)(1 + 2^2 + 3^2 + \ldots + n^2)} = \lim_{n\to\infty} \frac{n\left(\frac{n(n+1)}{2}\right)}{\frac{n(n+1)(2n+1)}{6}} = \frac{3}{2}$$

21. $$265\left[\lim_{h\to 0} \frac{h^2 + 3}{\left(\frac{f(1-h) - f(1)}{-h}\right)\left(\frac{\sin 5h}{h}\right)}\right] = -265 \times \frac{3}{f'(1)\cdot 5} = -\frac{53 \times 3}{f'(1)}$$

$$= -\frac{53 \times 3}{-53} \qquad [\because f'(1) = -53]$$

$$= 3$$

22. $\lim\limits_{x\to 0} \dfrac{\cos^2 x - 1}{\cos x \cdot x^2 \cdot (x + 1)}$

$$\lim_{x\to 0} -\left(\frac{\sin^2 x}{x^2}\right)\frac{-1}{\cos x(x + 1)} = -1$$

23. $f(x + y) = f(x)\cdot f(y)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h} = f(x)\left(\lim_{h\to 0} \frac{f(h) - 1}{h}\right)$$

If $f(h) = 1 + hP(h) + h^2Q(h) \Rightarrow f'(x) = f(x)\lim\limits_{h\to 0} \dfrac{hP(h) + h^2Q(h)}{h} = P(0)f(x)$

24. $$\lim_{x\to\frac{\pi}{2}}\frac{\left(1-\tan\frac{x}{2}\right)(1-\sin x)}{\left(1+\tan\frac{x}{2}\right)(\pi-2x)^3}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\left(1-\cos\left(\frac{\pi}{2}-x\right)\right)}{(\pi-2x)^3}$$

Let $x=\frac{\pi}{2}+h$

$$\lim_{x\to0}\frac{\tan\left(-\frac{h}{2}\right)(1-\cos h)}{(-2h)^3}=\frac{1}{32}$$

25. $$\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x=e^{\lim_{x\to\infty}x\left(\frac{-5}{x+2}\right)}=e^{-5}$$

27. $\ln c=I,\quad (I\in\text{integer})$

$\Rightarrow\quad c=e^I$

c is rational when $I=0$

28. $$\lim_{x\to0}\left(1+\frac{a\sin bx}{\cos x}\right)^{1/x}=e^{\lim_{x\to0}\frac{1}{x}\left[1+\frac{a\sin bx}{\cos x}-1\right]}=e^{ab}$$

30. $$a=\lim_{x\to1}\left(\frac{x}{\ln x}-\frac{1}{x\ln x}\right)=\lim_{x\to1}\frac{x^2-1}{x\ln x}=\lim_{x\to1}\frac{x+1}{x}\cdot\frac{x-1}{\ln x}=2$$

$b=-4, c=1, d=-2$

32. $$f(x)=\begin{cases}x^2 & -1<x<0\\ =1 & x=0\\ =\frac{1}{x^2} & 0<x<1\end{cases}$$

$$\lim_{x\to0^-}\{f(x)\}+\lim_{x\to1^-}\{f(x)\}+\lim_{x\to-1^-}\{f(x)\}=0$$

33. Let $\sin^{-1}x=\theta$

$$\Rightarrow\quad\lim_{\theta\to\frac{\pi}{4}^+}\frac{\cos^{-1}\sin2\theta}{\sin\theta-\sin\frac{\pi}{4}}=\lim_{\theta\to\frac{\pi}{4}^+}\frac{2\theta-\frac{\pi}{2}}{2\sin\left(\frac{\theta-\frac{\pi}{4}}{2}\right)\cos\left(\frac{\theta+\frac{\pi}{4}}{2}\right)}=2\sqrt{2}$$

$$\lim_{\theta\to\frac{\pi}{4}^-} \frac{\cos^{-1}\sin 2\theta}{\sin\theta-\sin\frac{\pi}{4}}=-2\sqrt{2}$$

34. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\sin\frac{\pi}{2k}-\sin\frac{\pi}{2(k+2)}\right)+\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\cos\frac{\pi}{2(k+2)}-\cos\frac{\pi}{2k}\right)$

$$=\lim_{n\to\infty}\left(\sin\frac{\pi}{2}-\sin\frac{\pi}{6}+\sin\frac{\pi}{4}-\sin\frac{\pi}{8}+\sin\frac{\pi}{6}-\sin\frac{\pi}{10}+\ldots+\sin\frac{\pi}{2n}-\sin\frac{\pi}{2(n+2)}\right)$$

$$+\lim_{n\to\infty}\left(\cos\frac{\pi}{6}-\cos\frac{\pi}{2}+\cos\frac{\pi}{8}-\cos\frac{\pi}{4}+\cos\frac{\pi}{10}-\cos\frac{\pi}{6}+\ldots+\cos\frac{\pi}{2(n+2)}-\cos\frac{\pi}{2n}\right)$$

$$=1+\frac{1}{\sqrt{2}}+2-\frac{1}{\sqrt{2}}=3$$

36. $\lim\limits_{x\to0}\frac{(\cos x)^{\frac{1}{m}-\frac{1}{n}}-1}{x^2}=\lim\limits_{x\to0}\frac{\left(1-2\sin^2\frac{x}{2}\right)^{\frac{1}{m}-\frac{1}{n}}-1}{x^2}$

$$=\lim_{x\to0}-2\left(\frac{1}{m}-\frac{1}{n}\right)\frac{\sin^2\frac{x}{2}}{x^2}=\frac{m-n}{2mn}$$

37. $\lim\limits_{x\to0}\frac{x+xa\cos x-b\sin x}{x^3}=1$

Using expansion,

$$\Rightarrow \lim_{x\to0}\frac{x+xa\left(1-\frac{x^2}{2!}\right)-b\left(x-\frac{x^3}{3!}\right)}{x^3}\Rightarrow\lim_{x\to0}\frac{x+ax-\frac{ax^3}{2!}-bx+\frac{bx^3}{3!}}{x^3}$$

Clearly, $1+a-b=0$ for limit to be finite ...(1)

$$\Rightarrow \lim_{x\to0}\left(\frac{b}{3!}-\frac{a}{2!}\right)\frac{x^3}{x^3}=1\Rightarrow\frac{b}{6}-\frac{a}{2}=1\Rightarrow b-3a=6 \quad\ldots(2)$$

$\Rightarrow$ From (1) and (2), $a=-\frac{5}{2},b=-\frac{3}{2}$

38. $\lim\limits_{x\to0}\frac{a\cos ax-\frac{e^x(\cos x-\sin x)}{e^x\cdot\cos x}}{\sin bx+bx\cdot\cos bx}=\frac{1}{2}$

$$\Rightarrow\lim_{x\to0}\frac{\cos^2x-\cos x+\sin x}{\cos x(\sin bx+bx\cos bx)}=\frac{1}{2}\qquad(\because a=1)$$

39. $\alpha=\lim\limits_{n\to\infty}\frac{(1^3+2^3+3^3\ldots+n^3)-(1^2+2^2\ldots+n^2)}{n^4}=\lim\limits_{n\to\infty}\frac{\left(\frac{n(n+1)}{2}\right)^2-\frac{n(2n+1)(n+1)}{6}}{n^4}$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{1}{4}\left(1+\frac{1}{n}\right)^2-\frac{(2n+1)(n+1)}{6n^3}\right]=\frac{1}{4}$$

40. $\lim\limits_{x\to0}\dfrac{\cos(\sin x)-\cos x}{x^4} \Rightarrow \lim\limits_{x\to0}\dfrac{2\sin\left(\frac{\sin x+x}{2}\right)\sin\left(\frac{x-\sin x}{2}\right)}{x^4}$

$$\Rightarrow \lim_{x\to0}\frac{2}{x^4}\frac{(\sin x+x)}{2}\cdot\frac{(x-\sin x)}{2} \qquad \left\{\because \frac{\sin x+x}{2}\to0;\ \frac{x-\sin x}{2}\to0\right\}$$

$$\Rightarrow \lim_{x\to0}\frac{1}{2}\left(1+\frac{\sin x}{x}\right)\left(\frac{x-\sin x}{x^3}\right)\Rightarrow\frac{1}{6}$$

42.

$$u_n=\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}\ldots+\frac{n}{2^n} \qquad \ldots(1)$$

$$\frac{1}{2}u_n=\ldots+\frac{1}{2^2}+\frac{2}{2^3}\ldots+\frac{n-1}{2^n}+\frac{n}{2^{n+1}} \qquad \ldots(2)$$

Substracting equation (1) and (2),

$$\frac{u_n}{2}=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}-\frac{n}{2^{n+1}} \Rightarrow \frac{u_n}{2}=\frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{\left(1-\frac{1}{2}\right)}-\frac{n}{2^{n+1}}$$

$$\Rightarrow u_n=2\left(1-\frac{1}{2^n}\right)-\frac{n}{2^{n+1}};\ \lim_{n\to0}u_n=2$$

43. $e^{\lim\limits_{x\to0}\frac{(\cos x-1)}{\sin^2x}}+\lim\limits_{x\to0}\dfrac{\frac{\sin 2x}{2x}\cdot2x+6x\cdot\frac{\tan^{-1}3x}{3x}+3x^2}{\frac{\ln(1+3x+\sin^2x)}{3x+\sin^2x}\cdot(3x+\sin^2x)+xe^x}=\dfrac{1}{\sqrt{e}}+2$

44. $\tan\dfrac{x}{2}(1+\sec x)=\tan x$

$$f_n(x)=\tan\frac{x}{2}(1+\sec x)(1+\sec 2x)\ldots(1+\sec 2^n x)=\tan 2^n x$$

45. $\lim\limits_{x\to\frac{\pi}{4}}(1+[x])^{\frac{1}{\ln(\tan x)}}=\lim\limits_{x\to\frac{\pi}{4}}(1)^{\frac{1}{\ln(\tan x)}}=1$

46. $\lim\limits_{x\to0}\dfrac{\{(a-n)nx-\tan x\}}{x^2}\dfrac{\sin nx}{nx}xn=0$

$$\Rightarrow \{(a-n)n-1\}n=0\Rightarrow a=n+\frac{1}{n}$$

47. $y=\lim\limits_{n\to\infty}\left(\dfrac{n!}{n^n}\right)^{\frac{3n^3+4}{4n^4-1}}$

$$\ln y=\lim_{n\to\infty}\frac{3n^3+4}{4n^4-1}\sum_{r=1}^{n}\ln\left(\frac{r}{n}\right)=\frac{3}{4}\int_0^1 \ln x\,dx=\frac{-3}{4}\Rightarrow y=e^{-3/4}$$

48. $\lim\limits_{x\to\infty}\dfrac{ax^2+bx+c}{dx+e}=\lim\limits_{x\to\infty}\dfrac{ax+b+(c/x)}{d+(e/x)}=\lim\limits_{x\to\infty}\left(\dfrac{a}{d}x+\dfrac{b}{d}\right)$

$=+\infty$ if $\left(\dfrac{a}{d}\right)$ is positive.

$=-\infty$ if $\left(\dfrac{a}{d}\right)$ is negative.

Alternate solution :

$$\lim_{x\to\infty}\frac{ax^2+bx+c}{dx+e}=\lim_{x\to\infty}\frac{a+(b/x)+(c/x^2)}{(d/x)+(e/x^2)}$$

Here $\dfrac{e}{x^2}<<\dfrac{d}{x}$. Therefore,

$$\lim_{x\to\infty}\frac{ax^2+bx+c}{dx+e}=\lim_{x\to\infty}\frac{a}{d/x}$$

$$=\begin{cases}\dfrac{a}{0^+}\text{ if } d>0 & \begin{cases}+\infty\text{ if } a>0\text{ and } d>0\\ -\infty\text{ if } a<0\text{ and } d>0\end{cases}\\ \dfrac{a}{0^-}\text{ if } d<0 & \begin{cases}-\infty\text{ if } a>0\text{ and } d<0\\ +\infty\text{ if } a<0\text{ and } d<0\end{cases}\end{cases}$$

49. $f(x)=\lim\limits_{n\to\infty}\tan^{-1}\left(4n^2\cdot 2\sin^2\dfrac{x}{2n}\right)=\lim\limits_{n\to\infty}\tan^{-1}\left(8n^2\left(\dfrac{\sin\frac{x}{2n}}{\frac{x}{2n}}\right)^2\cdot\dfrac{x^2}{4n^2}\right)=\tan^{-1}(2x^2)$

$$g(x)=\lim_{n\to\infty}\frac{n^2}{2}\left(\frac{\ln\left(1+\cos^2\frac{2x}{n}-1\right)}{\cos^2\frac{2x}{n}-1}\right)\left(\cos\frac{2x}{n}-1\right)=x^2$$

50. $\lim\limits_{x\to 0}\dfrac{\sin^2 x}{f(x)}=\dfrac{1}{3}\Rightarrow f(x)=x^2(ax+3);\quad a\neq 0$

51. $\lim\limits_{x\to 0}\dfrac{(2e^{2\sin x}-e^{\sin x}-1)}{(x^2+2x)\,e^{\sin x}}=\lim\limits_{x\to 0}\dfrac{(2e^{\sin x}+1)(e^{\sin x}-1)}{x(x+2)\,e^{\sin x}}=\dfrac{3}{2}$

52. $x^n + ax + b = (x - x_1)(x - x_2)(x - x_3)\dots(x - x_n)$

$$\lim_{x \to x_1} \frac{x^n + ax + b}{x - x_1} = (x_1 - x_2)(x_1 - x_3)\dots(x_1 - x_n)$$

53. $$\lim_{x \to 0} \frac{\left(1 + \frac{1}{3}\sin^2 x + \dots\right) - \left(1 - \frac{1}{4}(2\tan x) + \dots\right)}{\sin x + \tan^2 x} = \frac{1}{2}$$

54. $$\lim_{x \to 0} \frac{f(x)}{x^2} = \lim_{x \to 0} \begin{vmatrix} \cos x & \frac{2\sin x}{x} & \tan x \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -1$$

Exercise-2 : One or More than One Answer is/are Correct

1. $e^{\lim_{x \to 0} \frac{1}{3x^2}(p\tan qx^2 - 3\cos^2 x + 3)}$

$e^{\lim_{x \to 0} \frac{pq}{3} + \frac{3(1 - \cos^2 x)}{3x^2}}$

$\Rightarrow \quad \frac{pq}{3} + 1 = \frac{5}{3}; \qquad pq = 2$

3. $a \geq e > 2$

(a) $L = a \lim_{x \to \infty} \left(1 + \left(\frac{2}{a}\right)^x + \left(\frac{e}{a}\right)^x\right)^{1/x}$

$\therefore \quad x \to a, \left(\frac{2}{a}\right) \to 0, \left(\frac{e}{a}\right) \to 0, \frac{1}{x} \to 0$

So, $L = a$

(b) If $a = 2e > 2$

$$L = \lim_{x \to \infty} (2^x + (2e)^x + e^x)^{1/x} = 2e \lim_{x \to \infty} \left[\left(\frac{1}{e}\right)^x + 1 + \left(\frac{1}{2}\right)^x\right]^{1/x} = 2e(1) = 2e$$

(c) If $0 < a \leq e$

$$L = e\left(\lim_{x \to \infty} \left(\left(\frac{2}{e}\right)^x + \left(\frac{a}{e}\right)^x + 1\right)^{1/x}\right) = e$$

(d) $a > \frac{e}{2} > 1$

$$L = \lim_{x\to\infty}\left[2^x + \left(\frac{2a}{2}\right)^x + e^x\right]^{1/x} = 2a\lim_{x\to\infty}\left(\left(\frac{1}{a}\right)^x + \left(\frac{1}{2}\right)^x + \left(\frac{e}{2a}\right)^x\right)^{1/x} = 0$$

5. $f(x) = \cos(\sin x)$

Range is $[\cos 1, 1]$.

8. $f(x) = x\left(\frac{3}{2} + \frac{3}{2}[\cos x]\right)$

9. If $x \neq \frac{1}{2^{2^n}}$ then $f(x) = 0$ but if $x = \frac{1}{2^{2^n}}$ then $\lim\limits_{x\to 0} f(x) = \lim\limits_{n\to\infty}(-1)^n$, hence does not exist.

Also, if $x = \frac{1}{2^{2^n}}$ then $2x \neq \frac{1}{2^{2^n}} \Rightarrow f(2x) = 0$

11. $$\lim_{x\to 0^+}\frac{\cos^{-1}(1-x)\sin^{-1}(1-x)}{\sqrt{2x}\,(1-x)} = \lim_{x\to 0^+}\frac{\left(\frac{\sin^{-1}\sqrt{2x-x^2}}{\sqrt{2x-x^2}}\right)\sqrt{2x-x^2}\cdot\sin^{-1}(1-x)}{\sqrt{2x}(1-x)} = \frac{\pi}{2}$$

$$\lim_{x\to 0^-}\frac{\cos^{-1}(-x)\sin^{-1}(-x)}{\sqrt{2(x+1)}\,(-x)} = \frac{\pi}{2\sqrt{2}}$$

12. $$\lim_{x\to 0}\frac{2\sin\left(\frac{\sin x - x}{2}\right)\cdot\cos\left(\frac{\sin x + x}{2}\right)}{ax^3 + bx^5 + c} = \frac{-1}{12}$$

$$\lim_{x\to 0}\frac{2\left(\frac{\sin\left(\frac{\sin x - x}{2}\right)}{\frac{\sin x - x}{2}}\right)\left(\frac{\sin x - x}{2}\right)\cdot\cos\left(\frac{\sin x + x}{2}\right)}{ax^3 + bx^5 + c} = \frac{-1}{12}$$

14. $\cos^2\left(n\pi + \frac{\pi}{3}\right)$

15. $\sin\alpha + \sin\beta = -\frac{\sin\beta}{\sin\alpha} \Rightarrow \sin\alpha = \sin\beta = -\frac{1}{2}$

16. $\lim\limits_{x\to 2^+}[5 - 2x] = 0$

$\lim\limits_{x\to 2^-}[|x-2| + a^2 - 6a + 9] = 0 \Rightarrow (a-3)^2 < 1$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

1. $S_1 = 1, S_2 = 7, S_3 = 19$

$\Rightarrow \quad S_n = 1 + 3n(n-1)$

$$\lim_{n\to\infty} \frac{S_n}{n^2} = 3$$

2. $r_1 = 1, r_2 = \frac{1}{3}, r_3 = \frac{1}{5}$

or $\quad r_n = \frac{1}{2n-1}$

$$\lim_{n\to\infty} n \times \frac{1}{2n-1} = \frac{1}{2}$$

Paragraph for Question Nos. 3 to 4

3. $x > 0,\ x < \tan x$

$x < 0,\ x > \tan x \Rightarrow x - \tan x > 0$

$\therefore \quad [x - \tan x] = 0$

$$\therefore \quad \lim_{x\to0^-} f([x - \tan x]) = f(0) = 4$$

4. $x > 0 \qquad x < \tan x$

$$\frac{x}{\tan x} < 1$$

$$\lim_{x\to0^+} \left\{\frac{x}{\tan x}\right\} = \frac{x}{\tan x} \to 1^-$$

$$\therefore \quad \lim_{x\to0^+}\left(f\left\{\frac{x}{\tan x}\right\}\right) = \lim_{x\to0^+} f\left(\frac{x}{\tan x}\right) = f(1^-) = 2 + 5 = 7$$

Paragraph for Question Nos. 5 to 6

5. $f(x) = 1 - |x-2|$

$x \to 2^+, f(x) \to 1^-$ and $x \to 2^-, f(x) \to 1^-$

$$\text{R.H.L.} = \lim_{x\to2^+} (f(x))^{\frac{1}{\sin\left(\frac{\pi x}{2}\right)}} = e^{\lim\limits_{x\to2^+} \frac{f(x)-1}{\sin\left(\frac{\pi x}{2}\right)}}$$

$$= e^{\lim\limits_{x\to 2^+}\frac{1-(x-2)-1}{\sin\pi\left(1-\frac{x}{2}\right)}} = e^{\lim\limits_{x\to 2^+}\frac{(x-2)}{\left(\frac{\sin\frac{\pi}{2}(2-x)}{\frac{\pi}{2}(2-x)}\right)}\times\frac{\pi}{2}(2-x)}$$

$$= e^{2/\pi}$$

$$\text{L.H.L.} = \lim_{x\to 2^-}(f(x))^{\frac{1}{\sin x\left(\frac{\pi x}{2}\right)}} = e^{\lim\limits_{x\to 2^-}\frac{f(x)-1}{\sin\left(\frac{\pi x}{2}\right)}}$$

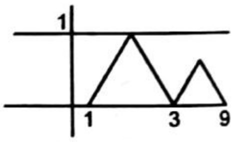
$$= e^{\lim\limits_{x\to 2^-}\frac{1+(x-2)-1}{\sin\frac{\pi}{2}(2-x)}} = e^{\lim\limits_{x\to 2^-}\frac{x-2}{\frac{\pi}{2}(2-x)}}$$

$$= e^{-2/\pi}$$

$\therefore$ Limit does not exist.

6. $[1,3]$

as $f(3x) = \alpha f(x)$



$x \in [1,3]$; $f(x) \in [0,1]$

$3x \in [3,9]$; $f(3x) = \alpha f(x) \in [0,\alpha]$

$9x \in [9,27]$; $f(9x) = \alpha f(3x) \in [0,\alpha^2]$

area between $[1,3]$ is $\Delta_1 = \frac{1}{2}\times 2\times 1 = 1$

area between $[3,9]$ is $\Delta_2 = \frac{1}{2}\times 6\times\alpha = 3\alpha$

area between $[9,27]$ is $\Delta_3 = \frac{1}{2}\times 18\times\alpha^2 = 9\alpha^2$

$\therefore$ $1, 3\alpha, 9\alpha^2, \ldots\ldots$ is converges when (g.p.) $|3\alpha| < 1$ $\alpha \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

Paragraph for Question Nos. 7 to 9

7. $$\lim_{x\to 0}\frac{[(1+bx)-(1+ax)\sqrt{1+x}]}{x^3} = \lim_{x\to 0}\frac{(1+bx)-(1+ax)\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}\right)}{x^3}$$

$$= \lim_{x\to 0}\frac{bx-\frac{x}{2}+\frac{x^2}{8}-\frac{x^3}{16}-ax-\frac{ax^2}{2}+\frac{ax^3}{8}}{x^3}$$

$\Rightarrow$ coefficient of x and $x^2 = 0 \Rightarrow b - a = \frac{1}{2}$ and $\frac{a}{2} = \frac{1}{8}$

$\Rightarrow \quad a = \frac{1}{4}, b = \frac{3}{4}$

8. $a + b = 1$

9. $l = -\frac{1}{32}$; $b = \frac{3}{4}$

Paragraph for Question Nos. 10 to 11

Sol. $\sin x + \sin y = 1$

$$y' = \frac{-\cos x}{\sqrt{2\sin x - \sin^2 x}}$$

$$\Rightarrow \quad y'' = \frac{\sin^2 x - \sin x + 1}{(2\sin x - \sin^2 x)^{3/2}}$$

Exercise-5 : Subjective Type Problems

1. $$\lim_{x\to 0} - \frac{\ln\tan\left(\frac{\pi}{4} - \beta x\right)}{\tan\alpha x} = -\lim_{x\to 0} \frac{\ln\left[\left(\frac{1-\tan\beta x}{1+\tan\beta x} - 1\right) + 1\right]}{\tan\alpha x}$$

$$= -1\left(-2\frac{\beta}{\alpha}\right) = 1$$

$$\Rightarrow \quad \frac{\alpha}{\beta} = 2$$

3. $a(x^3 - 1) + (x - 1) = 0$

$(x - 1)(ax^2 + ax + a + 1) = 0$

$\alpha, \beta \neq 1$ so, α, β are roots of $ax^2 + ax + a + 1 = 0$

$\alpha + \beta = -1, \ \alpha\beta = \frac{a+1}{a}$

$$\lim_{x\to\frac{1}{\alpha}} \frac{(1+a)x^3 - x^2 - a}{(e^{1-\alpha x} - 1)(x - 1)} = \lim_{x\to\frac{1}{\alpha}} \frac{(x^3 - x^2) + a(x^3 - 1)}{(e^{1-\alpha x} - 1)(x - 1)}$$

$$= \lim_{x\to\frac{1}{\alpha}} \frac{[x^2 + a(x^2 + x + 1)]}{(e^{1-\alpha x} - 1)} = \lim_{x\to\frac{1}{\alpha}} \frac{(1+a)x^2 + ax + a}{\left(\frac{e^{1-\alpha x} - 1}{1 - \alpha x}\right)(1 - \alpha x)}$$

$$= \lim_{x\to\frac{1}{\alpha}} a\frac{\left[\left(\frac{1+a}{a}\right)x^2+(1)x+1\right]}{(1-\alpha x)} = \lim_{x\to\frac{1}{\alpha}} a\frac{(\alpha\beta x^2-(\alpha+\beta)x+1)}{(1-\alpha x)}$$

$$= \lim_{x\to\frac{1}{\alpha}} a\frac{(1-(\alpha)x)(1-(\beta)x)}{(1-\alpha x)} = \frac{a(\alpha-\beta)}{\alpha}$$

4. $\lim_{x\to 0}\frac{(4^x-1)(5^x-1)(7^x-1)}{x\sin^2 x} = 2\ln 2\ln 5\ln 7$

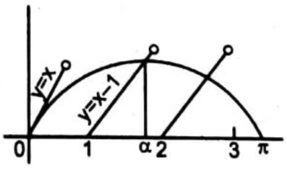
5. $\lim_{x\to 0}\frac{ax\cos x+b\sin x}{x^2\sin x} = \frac{1}{3}$

$$\lim_{x\to 0}\frac{ax\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\ldots\right)+b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\ldots\right)}{x^2\sin x} = \frac{1}{3}$$

$a+b=0$ and $-\frac{a}{2}-\frac{b}{6}=\frac{1}{3}$

$\Rightarrow b=1,\ a=-1$

7. $\lim_{x\to\alpha^+}\left[\frac{\sin x}{x-1}\right]=0$



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Chapter 3 – Continuity, Differentiability, and Differentiation

3

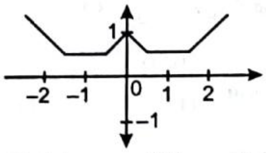
CONTINUITY, DIFFERENTIABILITY AND DIFFERENTIATION

Exercise-1 : Single Choice Problems

1. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x) + f(h) + 3hx(h+x) - f(x)}{h}$

$f'(x) = 3x^2 + f'(0) \Rightarrow f''(x) = 6x$

2.



$f(x)$ is non-differentiable at five points.

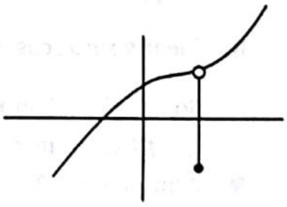
3. $\frac{x}{5}$ is integer at 21 points in $[0, 100]$

$\frac{x}{2}$ is integer at 51 points in $[0, 100]$

$\therefore$ But when x is a multiple of 10 then $f(x)$ is continuous.

So that respective points should be subtract from both *i.e.*, multiple of 10 are 11 points in $[0, 100]$.

$$21 + 51 - 11 - 11 = 72 - 22 = 50$$



4. $f(x)$ has isolated point of discontinuity but $|f(x)|$ is continuous at $x = a$

So, $\lim_{x \to a} f(x)$ and $f(a)$ has opposite sign, with same magnitude.

So, $\lim_{x \to a} f(x) = -f(a)$

$\lim_{x \to a} f(x) + f(a) = 0$

5. $\lim_{x \to 0} \frac{f(4x) - 3f(3x) + 3f(2x) - f(x)}{x^3} = 12$

$$\lim_{x\to 0}\frac{4f'(4x)-9f'(3x)+6f'(2x)-f'(x)}{3x^2}=12$$

$$\lim_{x\to 0}\frac{4^2f''(4x)-27f''(3x)+12f''(2x)-f''(x)}{6x}=12$$

$$\lim_{x\to 0}\frac{4^3f'''(4x)-81f'''(3x)+24f'''(2x)-f'''(x)}{6}=12$$

$$\therefore \quad (4^3-81+24-1)\,f'''(0)=12\times 6$$

$$6f'''(0)=12\times 6$$

$$f'''(0)=12$$

6. $$y=\frac{1}{1+(\tan\theta)^{\sin\theta-\cos\theta}+(\tan\theta)^{\cot\theta-\cos\theta}}+\frac{1}{1+(\tan\theta)^{\cos\theta-\sin\theta}+(\tan\theta)^{\cot\theta-\sin\theta}}+\frac{1}{1+(\tan\theta)^{\cos\theta-\cot\theta}+(\tan\theta)^{\sin\theta-\cot\theta}}$$

$$y=\frac{(\tan\theta)^{\cos\theta}}{(\tan\theta)^{\cos\theta}+(\tan\theta)^{\sin\theta}+(\tan\theta)^{\cot\theta}}+\frac{(\tan\theta)^{\sin\theta}}{(\tan\theta)^{\cos\theta}+(\tan\theta)^{\sin\theta}+(\tan\theta)^{\cot\theta}}+\frac{(\tan\theta)^{\cot\theta}}{(\tan\theta)^{\cos\theta}+(\tan\theta)^{\sin\theta}+(\tan\theta)^{\cot\theta}}$$

$$y=1$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3}=0$$

7. $f'(x)=\sin(x^2)$

$$y=f(x^2+1)$$

$$\frac{dy}{dx}=f'(x^2+1)2x$$

$$\left.\frac{dy}{dx}\right|_{x=1}=2\cdot f'(2)=2\sin 4$$

8. Clearly $\sin x, \cos x$ are negative at $x=\dfrac{7\pi}{6}$

So, $f(x)=-(\sin x+\cos x)$

$f'(x)=(\sin x-\cos x)$

9. $2\sin x\cos y=1$

$\cos x\cos y-\sin x\sin y\cdot y'=0 \;\Rightarrow\; y'_{(\pi/4,\,\pi/4)}=1$

$y'=\cot x\cot y$

$y''=-\cot x\operatorname{cosec}^2 y\times y'-\cot y\operatorname{cosec}^2 x$

$y''_{(\pi/4,\,\pi/4)}=-(1\times 2\times 1)-(1\times 2)=0$

10. $\frac{dx}{dt}=2t\,f'(t^2),\quad \frac{dy}{dt}=3t^2f'(t^3)$

$$\frac{dy}{dx}=\frac{3}{2}\frac{tf'(t^3)}{f'(t^2)}$$

$$\frac{d^2y}{dx^2}=\frac{3}{2}\left(\frac{f'(t^2)(f'(t^3)+3t^3f''(t^3))-2t^2f'(t^3)\cdot f''(t^2)}{(f'(t^2))^2}\right)\frac{dt}{dx}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=1}=\frac{3}{2}\left(\frac{f'(1)(f'(1)+3f''(1))-2f'(1)\cdot f''(1)}{(f'(1))^2}\right)\frac{1}{2f'(1)}=\frac{3}{4}\left(\frac{f''(1)+f'(1)}{(f'(1))^2}\right)$$

11. L.H.L. $=a+1$

R.H.L. $=b+1$

$\therefore$ they are continuous L.H.L. = R.H.L.

12. $$y=\frac{\frac{1}{x}}{\frac{1}{x}-\alpha}+\frac{\frac{\beta}{x}}{\left(\frac{1}{x}-\alpha\right)\left(\frac{1}{x}-\beta\right)}+\frac{\frac{\gamma}{x^2}}{\left(\frac{1}{x}-\alpha\right)\left(\frac{1}{x}-\beta\right)\left(\frac{1}{x}-\gamma\right)}$$

$$=\frac{\frac{1}{x^2}}{\left(\frac{1}{x}-\alpha\right)\left(\frac{1}{x}-\beta\right)}+\frac{\frac{\gamma}{x^2}}{\left(\frac{1}{x}-\alpha\right)\left(\frac{1}{x}-\beta\right)\left(\frac{1}{x}-\gamma\right)}=\frac{\frac{1}{x^3}}{\left(\frac{1}{x}-\alpha\right)\left(\frac{1}{x}-\beta\right)\left(\frac{1}{x}-\gamma\right)}$$

$$\log y=-3\ln x-\ln\left(\frac{1}{x}-\alpha\right)-\ln\left(\frac{1}{x}-\beta\right)-\ln\left(\frac{1}{x}-\gamma\right)$$

$$\frac{1}{y}y'=\frac{-3}{x}+\frac{\frac{1}{x^2}}{\left(\frac{1}{x}-\alpha\right)}+\frac{\frac{1}{x^2}}{\left(\frac{1}{x}-\beta\right)}+\frac{\frac{1}{x^2}}{\left(\frac{1}{x}-\gamma\right)}$$

$$y'=\frac{y}{x}\left(-3+\frac{\frac{1}{x}}{\left(\frac{1}{x}-\alpha\right)}+\frac{\frac{1}{x}}{\left(\frac{1}{x}-\beta\right)}+\frac{\frac{1}{x}}{\left(\frac{1}{x}-\gamma\right)}\right)$$

$$y'=\frac{y}{x}\left(\frac{\alpha}{1/x-\alpha}+\frac{\beta}{1/x-\beta}+\frac{\gamma}{1/x-\gamma}\right)$$

13. $f(x)=\sqrt{\frac{1+\sin^{-1}x}{1-\tan^{-1}x}}$

$$\ln f(x)=\frac{1}{2}[\ln(1+\sin^{-1}x)-\ln(1-\tan^{-1}x)]$$

$$\frac{f'(x)}{f(x)}=\frac{1}{2}\left[\frac{1}{(1+\sin^{-1}x)\sqrt{1-x^2}}+\frac{1}{(1-\tan^{-1}x)(1+x^2)}\right]$$

$\therefore \quad f'(0)=1$

14. $\sin^2 x=-\sin^2 x \Rightarrow 2\sin^2 x=0 \Rightarrow x=n\pi$

15. $$f(x)\begin{cases}\rightarrow \tan x & \tan x<\cot x\\ \rightarrow \cot x & \tan x\ge\cot x\end{cases}$$

Points of non-derivability $=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

16. $g(x)=|||x-1|-1|-1|$

$=x-3 \qquad x>3$

$=-(x-3) \qquad 2<x<3$

18. $$\frac{d^2x}{dy^2}=-\frac{1}{\left(\frac{dy}{dx}\right)^3}\frac{d^2y}{dx^2}$$

$$\frac{dy}{dx}=1+e^x,\ \frac{d^2y}{dx^2}=e^x$$

at $x=\ln 2,\ \frac{dy}{dx}=3,\ \frac{d^2y}{dx^2}=2$

$$\frac{d^2x}{dy^2}=\frac{-2}{27}$$

19. $$g'(f(x))=\frac{1}{f'(x)}$$

$f(x)=-4$ at $x=-2$

$$\Rightarrow \quad g'(-4)=\frac{1}{f'(-2)}=\frac{1}{2}$$

20. $f(x)=2-x \qquad x\ge 1$

$=x \qquad 0\le x<1$

$=-x \qquad -1\le x<0$

$=x+2 \qquad x<-1$

21. $f(x)=\cos x^2$

$f'(x)=-2x\sin x^2$

22. $f(g(x))=x \Rightarrow f'(g(x))g'(x)=1 \Rightarrow g'(x)=\frac{1}{f'(g(x))}=1+(g(x))^5$

$g''(x)=5(g(x))^4 g'(x)$

23. $f(x) = x^2 \qquad x \ge 1$

$= x \qquad 0 \le x \le 1$

$= 2x \qquad -1 \le x \le 0$

$= x - 1 \qquad x \le -1$

Clearly it is non-differentiable at $x = 0, -1$ and 1.

24. $f(x) = \lim\limits_{n\to\infty}\left(\cos\dfrac{x}{2}\cdot\cos\dfrac{x}{2^2}\cdot\cos\dfrac{x}{2^3}\ldots\ldots\cos\dfrac{x}{2^n}\right) = \lim\limits_{n\to\infty}\dfrac{\sin x}{2^n\sin\left(\dfrac{x}{2^n}\right)} = \dfrac{\sin x}{x}$

25. $f\left(\dfrac{\pi^-}{4}\right) = f\left(\dfrac{\pi}{4}\right) = f\left(\dfrac{\pi^+}{4}\right)$

$$\lim_{x\to\frac{\pi}{4}}\left(\frac{1-\tan x}{4x-\pi}\right) = \lim_{x\to\frac{\pi}{4}}\frac{\tan\left(\frac{\pi}{4}-x\right)\cdot(1+\tan x)}{4\left(x-\frac{\pi}{4}\right)} = -\frac{1}{2}$$

26. $f'(0) = \lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0}\dfrac{e^{-\frac{1}{h^2}}\sin\frac{1}{h}}{h}$

27. $\dfrac{dy}{dx} = 2y + 10$

$\displaystyle\int\frac{dy}{y+5} = 2\int dx$

$\ln(y+5) = 2x + c$

$y = 5(e^{2x} - 1) \qquad (\because c = \ln 5)$

$f(x) + 5\sec^2 x = 0 \Rightarrow e^{2x} + \tan^2 x = 0$

28. $f\left(\dfrac{\pi^-}{2}\right) = \lim\limits_{x\to\frac{\pi^-}{2}}\dfrac{\sin\{\cos x\}}{x-\frac{\pi}{2}} = \lim\limits_{x\to\frac{\pi^-}{2}}\dfrac{\sin(\cos x)}{x-\frac{\pi}{2}} = -1$

$f\left(\dfrac{\pi^+}{2}\right) = \lim\limits_{x\to\frac{\pi^+}{2}}\dfrac{\sin\{\cos x\}}{x-\frac{\pi}{2}} = \lim\limits_{x\to\frac{\pi^+}{2}}\dfrac{\sin(\cos x+1)}{x-\frac{\pi}{2}}$

29. Let $g(x) = f(e^x)$

$g'(x) = f'(e^x)\cdot e^x$

$g''(x) = f''(e^x)\cdot e^{2x} + f'(e^x)e^x$

30. $e^{f(x)} = \ln x \Rightarrow f(x) = \ln(\ln x) \Rightarrow g(x) = f^{-1}(x) = e^{e^x}$

$g'(x) = e^{e^x}\cdot e^x = e^{e^x + x}$

32. $\ln f(x) = 4\ln(x-1) + 3\ln(x-2) + 2\ln(x-3)$

$$\frac{f'(x)}{f(x)} = \frac{4}{x-1} + \frac{3}{x-2} + \frac{2}{x-3}$$

$$f'(x) = f(x)\left(\frac{4}{x-1} + \frac{3}{x-2} + \frac{2}{x-3}\right)$$

34. $f(2^+) = 0 \Rightarrow c = 0$

$$f(2^-) = \frac{b\sin\{-x\}}{\{-x\}} = f(2^+) = 0 \Rightarrow b = 0$$

35. $f(0) = \lim\limits_{x\to 0} \dfrac{e^{\tan x} - e^x + \ln(\sec x + \tan x) - x}{\tan x - x}$

$$= \lim_{x\to 0} e^x \frac{(e^{\tan x - x} - 1)}{\tan x - x} + \lim_{x\to 0} \frac{\ln(\sec x + \tan x) - x}{\tan x - x} = 1 + \lim_{x\to 0} \frac{\sec x - 1}{\sec^2 x - 1} = 1 + \frac{1}{2} = \frac{3}{2}$$

36. $f(0^-) = e^a$

$f(0) = b$

$c = 1$

$f(0^+) = \dfrac{2}{3} \Rightarrow b = e^a = \dfrac{2}{3}$

37. $\sqrt{x+y} + \sqrt{y-x} = 5$

$\sqrt{x+y} = 5 - \sqrt{y-x}$

Sq. both sides,

$\Rightarrow \quad x + y = 25 + y - x - 10\sqrt{y-x}$

$\Rightarrow \quad 25 - 2x = 10\sqrt{y-x}$

$\Rightarrow \quad -2 = \dfrac{10(y'-1)}{2\sqrt{y-x}}$

$\Rightarrow \quad -2\sqrt{y-x} = 5(y'-1)$

$\Rightarrow \quad -\left(5 - \dfrac{2x}{5}\right) = 5(y'-1)$

$\quad\quad -5 + \dfrac{2x}{5} = 5(y'-1)$

$\Rightarrow \quad y'' = \dfrac{2}{25}$

38. $g(x) = f^{-1}(x)$

$\Rightarrow \quad f(g(x)) = x$

$\Rightarrow \quad f'(g(x))g'(x) = 1$

$$\Rightarrow \quad g'(2) = \frac{1}{f'(g(2))}$$

$$f(1) = 2$$

$$\Rightarrow \quad g(2) = 1$$

$$\Rightarrow \quad g'(2) = \frac{1}{f'(1)}$$

$$f'(x) = 3x^2 + 4x^3 + \frac{1}{x}$$

$$f'(1) = 8$$

$$\Rightarrow \quad g'(2) = \frac{1}{8}$$

39. $f(x) = |x| \qquad x \in (-\infty, -1)$

$= x^2 \qquad x \in [-1, 1)$

$= 2x - 1 \qquad x \in [1, \infty)$

Function is not differentiable at $x = -1$.

40. $g(x) = (f(x))^2 + (f'(x))^2 \Rightarrow g'(x) = 2f(x)f'(x) + 2f'(x)f''(x)$

or $g'(x) = 2f(x)f'(x) - 2f(x)f'(x) = 0 \Rightarrow g(x) = c \Rightarrow g(8) = 8$

41. $$l = \lim_{x\to\infty}\left(f\left(\frac{a}{\sqrt{x}}\right)\right)^x = e^{\lim\limits_{x\to\infty}\left(\frac{f\left(\frac{a}{\sqrt{x}}\right)-1}{\frac{1}{x}}\right)}$$

Using L Hospital's rule, we get

$$l = e^{\frac{a^2}{2}f''(0)} = e^{-\frac{a^2}{2}}$$

42. $$\frac{d}{dx}f_n(x) = e^{f_{n-1}(x)}\frac{d}{dx}f_{n-1}(x) = f_n(x)\frac{d}{dx}f_{n-1}(x)$$

$$= f_n(x)f_{n-1}(x)\ldots\ldots f_2(x)f_1(x)$$

43. $y = \tan^{-1}(x^{1/3}) - \tan^{-1}(a^{1/3})$

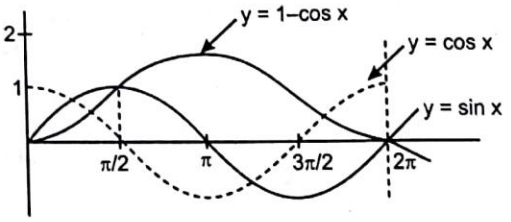
44. $f(x)$ is continuous at $x = 0$ then $\dfrac{4k-1}{3} = \dfrac{4k+1}{5}$

45. Put $x = \sin\theta$ then $y = \tan^{-1}\tan\dfrac{\theta}{2}$

46. $$\lim_{x\to 0}\frac{e^x\cos x - \ln(1+x) - 1}{x}$$

$$\lim_{x\to 0}\frac{(e^x-1)}{x}\cos x - \frac{\ln(1+x)}{x} + \left(\frac{\cos x - 1}{x}\right) = 0$$

47.



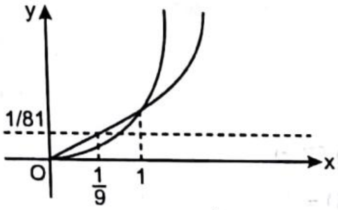
Clearly 3 sharp points.

48. $g(x) = f^{-1}(x)$ $\qquad f(4) = 2 \Rightarrow g(2) = 4$

$G(x) = \dfrac{1}{g(x)}$ $\qquad f'(4) = \dfrac{1}{16} \Rightarrow g'(2) = 16$

$$G'(x) = \frac{-1}{(g(x))^2}\cdot g'(x) \Rightarrow G'(2) = \frac{-1}{(g(x))^2}\cdot g'(2) = \frac{-1}{16}\cdot 16 = -1$$

49.



$$f(x) = \text{maximum}\left(x^4, x^2, \frac{1}{81}\right) = \frac{1}{81} \qquad x \le \frac{1}{9}$$

$$= x^2 \qquad \frac{1}{9} < x < 1$$

$$= x^4 \qquad x \ge 1$$

$f(x)$ is non-differentiable at $x = \dfrac{1}{9}, 1$

50. $\displaystyle\lim_{h\to 0} \frac{\ln(f(2+h^2)) - \ln(f(2-h^2))}{h^2}$

Apply L Hospital rule,

$$\lim_{h\to 0} \frac{\dfrac{2hf'(2+h^2)}{f(2+h^2)} + \dfrac{2hf'(2-h^2)}{f(2-h^2)}}{2h} = 4$$

51. $f(x) = (x^2 - 3x + 2)\,|(x-1)(x-2)(x-3)| + \left|\sin\left(x + \dfrac{\pi}{4}\right)\right|$

Not differentiable at $x = 3, \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

52. $h(x) = f(2x\,g(x) + \cos \pi x - 3)$

$h'(x) = f'(2x\,g(x) + \cos \pi x - 3)\,[2g(x) + 2xg'(x) - \pi \sin \pi x]$

$h'(1) = f'(2g(1) - 4)\,[2g(1) + 2g'(1)] = 32$

53. $f(x) = \dfrac{(x+1)^7\sqrt{1+x^2}}{(x^2-x+1)^6}$ $\qquad (f(0)=1)$

$$\ln f(x) = 7\ln(1+x) + \frac{1}{2}\ln(1+x^2) - 6\ln(x^2-x+1)$$

$$\frac{f'(x)}{f(x)} = \frac{7}{1+x} + \frac{x}{1+x^2} - \frac{6(2x-1)}{x^2-x+1}$$

$f'(0) = 13$

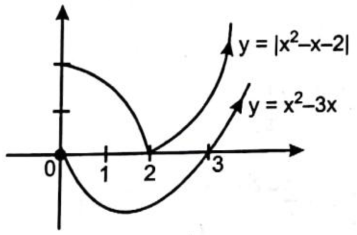
54. $f(x)\begin{cases} -\sin 2x; & x>1 \\ \ln(1+x); & x<1 \\ \dfrac{\ln 2-\sin 2}{2}; & x=1 \end{cases}$; $\qquad f(1^+) \neq f(1^-) \neq f(1)$

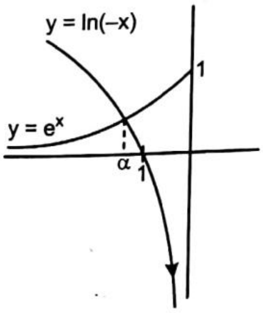
55. $f(f(x))\begin{bmatrix} f(x); & \text{if } f(x) \text{ is rational} \\ 1-f(x); & \text{if } f(x) \text{ is irrational} \end{bmatrix.$

$f(f(x))\begin{bmatrix} x; & \text{if } x \text{ is rational} \\ 1-(1-x); & \text{if } x \text{ is irrational} \end{bmatrix.$

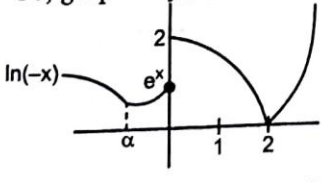
$f(f(x))\begin{bmatrix} x; & \text{if } x \text{ is rational} \\ x; & \text{if } x \text{ is irrational} \end{bmatrix.$

56.





So, graph of $f(x)$ is



Clearly, 3 non-differentiability points.

57. $g(f(x)) = x$

58. $$\lim_{x\to 0^-} \frac{\ln(2-\cos 2x)}{\ln^2(1+\sin 3x)} = K = \lim_{x\to 0^+} \frac{e^{\sin 2x}-1}{\ln(1+\tan 9x)}$$

$$\lim_{x\to 0^-} \frac{1-\cos 2x}{\sin^2 3x} = K = \lim_{x\to 0^+} \frac{\sin 2x}{\tan 9x}$$

59. $$\frac{dx}{dt} = -\frac{3}{t^4} - \frac{2}{t^3} = \frac{-3-2t}{t^4}$$

$$\frac{dy}{dt} = \frac{-3}{t^3} - \frac{2}{t^2} = \frac{-3-2t}{t^3}$$

$$\frac{dy}{dx} = t$$

$$\frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^3 = t - \left(\frac{1+t}{t^3}\right)\cdot t^3 = -1$$

60. $$-\frac{2}{y^3}y' = 2\sqrt{2}(-2\sin 2x)$$

$$\frac{(y')^2}{y^6} = 8-(2\sqrt{2}\cos x)^2 = 8-\left(\frac{1}{y^2}-1\right)^2$$

$$\frac{(y')^2}{y^6} = \frac{8y^2-(1-y^2)^2}{y^4}$$

$(y')^2 = 8y^4 - y^2(1-y^2)^2$ then diff.

61. $f(x) = x$ satisfy the equation.

$\therefore \quad f(5) = 5$

62. $$f(x)\left[\begin{array}{ll} x & x \le 0 \\ x^2 & 0 < x < 1 \\ 2x-1 & x \ge 1 \end{array}\right.$$

$$f'(x)\left[\begin{array}{ll} 1 & x \le 0 \\ 2x & 0 < x < 1 \\ 2 & x \ge 1 \end{array}\right.$$

$f(x)$ is not derivable at $x = 0$.

63. $y = (x+\sqrt{1+x^2})^n$

$$\frac{dy}{dx} = \frac{ny}{\sqrt{1+x^2}}$$

$$\frac{d^2y}{dx^2}=n\left[\frac{\sqrt{1+x^2}\,y'-\dfrac{yx}{\sqrt{1+x^2}}}{1+x^2}\right] \Rightarrow (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2y$$

64. $g'(x)=f'(x-\sqrt{1-x^2})\cdot\left(1+\dfrac{x}{\sqrt{1-x^2}}\right)=\left(1-\left(x-\sqrt{1-x^2}\right)^2\right)\cdot\left(\dfrac{x+\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)$

$$=2x\left(x+\sqrt{1-x^2}\right)$$

66. $f'(x)=\lim\limits_{h\to0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to0}f(x)\left(\dfrac{f(h)-1}{h}\right)=f(x)\cdot f'(0)=3f(x)$ $\quad(\because\ f'(0)=3)$

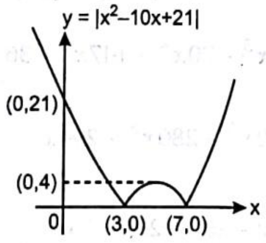
67. $f(x)=\lim\limits_{n\to\infty}\dfrac{\log_e(2+x)-x^{2n}\sin x}{1+x^{2n}}$

$$f(x)=\begin{cases}\ln(2+x) & |x|<1\\ -\sin x & |x|>1\\ \dfrac{\ln 3-\sin 1}{2} & x=1\\ \dfrac{\sin 1}{2} & x=-1\end{cases}$$

68. $\lim\limits_{x\to0}\dfrac{x-e^x+1-\{1-\cos 2x\}}{x^2}\Rightarrow\lim\limits_{x\to0}\dfrac{x-e^x+1-1+\cos 2x}{x^2}$

$$\Rightarrow \lim_{x\to0}\frac{x-e^x+\cos 2x}{x^2}=\lim_{x\to0}\frac{1+x-e^x}{x^2}+\frac{(\cos 2x-1)}{x^2}=-\frac{5}{2}$$

69.



71. $xy=\text{const.}$

$y+xy'=0\Rightarrow y'=-\dfrac{y}{x}$

72. $f(x)=-1+|x-2|$ is a continuous function.

$g(x) = 1 - |x|$ is a continuous function.

$\Rightarrow$ $f(g(x))$ is a continuous function.

73. $$f'(K^+) = \lim_{h \to 0} \frac{f(k+h) - f(k)}{h}$$

$$= \lim_{h \to 0} \frac{K \tan(\pi k + \pi h) - k \tan k\pi}{h}$$

$$= \lim_{h \to 0} k\left(\frac{\tan \pi h}{h}\right) = k\pi$$

74. $$\lim_{x \to 0} \frac{ae^{\sin x} + be^{-\sin x} - c}{x^2} = 2$$

$$a + b - c = 0 \qquad \ldots(1)$$

Applying L Hospital Rule,

$$\lim_{x \to 0} \frac{ae^{\sin x} \cdot \cos x - be^{-\sin x} \cdot \cos x}{2x} = 2 \Rightarrow a = b$$

75. $\tan x = \sec \alpha \cdot \tan y$

$\sec^2 x = \sec \alpha \cdot \sec^2 y \cdot y'$

$$y' = 1 \text{ at } \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$$

$$2 \sec^2 x \tan x = \sec \alpha (\sec^2 y \cdot y'' + 2 \sec^2 y \cdot \tan y \cdot (y')^2) \Rightarrow y'' = 0$$

76. We gave, $$y = (x^2 - 9)(x^2 - 4)(x^2 - 1)\, x$$

$$= \{x^6 - 14x^4 + x^2(49) - 36\}\, x$$

$$= x^7 - 14x^5 + 49x^3 - 36x$$

Therefore, $$\frac{dy}{dx} = 7x^5 - 70x^4 + 147x^2 - 36$$

Thus, $$\frac{d^2y}{dx^2} = 42x^5 - 280x^3 + 294x$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 42 - 280 + 294 = 56$$

77. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x)\, f(h) - f(x)}{h} = \lim_{h \to 0} f(x)\left(\frac{f(h) - 1}{h}\right)$$

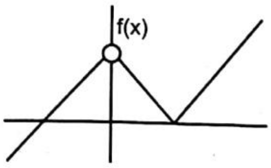
$\Rightarrow$ $f'(x) = f'(0),\ f(x) \Rightarrow f(x) = e^{kx}$ (where $k = f'(0)$)

78. $f(g(x)) = x\,;\qquad f'(g(x))g'(x) = 1 \Rightarrow g'(6) = \dfrac{1}{f'(g(6))} = \dfrac{1}{f'(0)}$

79. $\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx} = \dfrac{f'(x)}{g'(x)}$

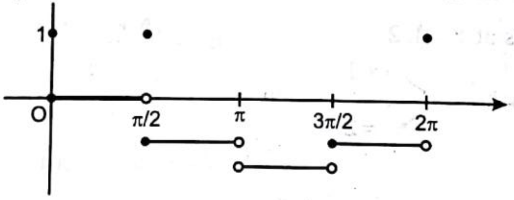
$$\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) = \frac{\dfrac{d}{dx}\left(\dfrac{dy}{dz}\right)}{\dfrac{dz}{dx}} = \frac{g'f'' - f'g''}{(g')^3}$$

80.



$$g(f(x)) = \begin{cases} f(x) + 1 = x + 2, x \in (-\infty, -1)\ x = -1, 1 \text{non differentiable} \\ (f(x) - 1)^2 = (x + 1 - 1)^2 = x^2, x \in (-1, 0) \\ (|x - 1| - 1)^2, x \geq 0 \end{cases}$$

81. $f(x) = [\sin x] + [\cos x]$



82. $g(x) = \begin{cases} \cos x & , x \in [0, \pi] \\ \sin x - 1 & , x > \pi \end{cases}$

$g(\pi^-) = g(\pi) = g(\pi^+) = -1$

but not differentiable at $x = \pi$.

83. $\displaystyle\sum_{r=0}^{\infty} \frac{f^r(0)}{r!} = \frac{f(0)}{0!} + \frac{f'(0)}{1!} + \frac{f''(0)}{2!} + \ldots$

$$= 4^n + \frac{n \cdot 4^{n-1}}{1!} + \frac{n(n-1) \cdot 4^{n-2}}{2!} + \ldots$$

$$= {}^nC_0 4^n + {}^nC_1 4^{n-1} + {}^nC_2 4^{n-2} + \ldots$$

$$= (4 + 1)^n = 5^n$$

84. $f(x) = \dfrac{x}{1-x} \qquad x \le -1$

$= \dfrac{x}{1+x} \qquad -1 < x < 0$

$= \dfrac{x}{1-x} \qquad 0 \le x < 1$

$= \dfrac{x}{1+x} \qquad x \ge 1$

Function is discontinuous at $x = -1, 1$

$f(x)$ is not differentiable at $x = -1, 1$

85. $f(g(x)) = x$

$$f'(g(x))g'(x) = 1 \Rightarrow g'\left(\frac{-7}{6}\right) = \frac{1}{f'\left(g\left(\frac{-7}{6}\right)\right)} = \frac{1}{f'(1)}$$

86. $f(x) = 0 \qquad x \ge 0$

$= 4x^2(1-2x)^2 \qquad x < 0$

Differentiable everywhere.

88. $f(x)$ is discontinuous at $x = 1, 2$

$$\Rightarrow \quad g(x) = x^2 - ax + b = 0 \begin{array}{l} \nearrow x=1 \\ \searrow x=2 \end{array}$$

89. $f^{-1}(f(x)) = x$

$(f^{-1}(f(x)))'f'(x) = 1$

$(f^{-1}(f(9)))'f'(g) = 1$

$(f^{-1}(3))' = \dfrac{1}{f'(9)} = \dfrac{1}{5}$

90. $f(0^+) = \lim\limits_{h \to 0} h^n \sin\dfrac{1}{h} = 0 \Rightarrow n > 0$

$f(0^-) = \lim\limits_{h \to 0} (-h)^n \sin\left(-\dfrac{1}{h}\right) = 0 \Rightarrow n > 0$

$f'(x) = n \cdot x^{n-1} \cdot \sin\dfrac{1}{x} - x^{n-2} \cdot \cos\dfrac{1}{x} = \text{finite} \Rightarrow n = 2$

Exercise-2 : One or More than One Answer is/are Correct

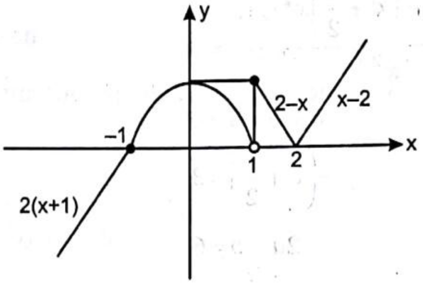
1. $f(x)$ has exactly one point of discontinuity so that sgn $(x^2 - \lambda x + 1)$ is equal to zero for some values of λ.

$\therefore \quad D = 0$

$\Rightarrow \quad \lambda^2 - 4 = 0$

$\Rightarrow \quad \lambda = \pm 2$

2. Answer from the graph.



3. (a) L.H.L. $= \lim\limits_{x\to 0^-} \dfrac{x(3e^{1/x}+4)}{2-e^{1/x}} = 0\left(\dfrac{4}{2}\right) = 0$

R.H.L. $= \lim\limits_{x\to 0^+} \dfrac{x(3e^{1/x}+4)}{2-e^{1/x}} = \lim\limits_{x\to 0^+} x\left(\dfrac{3+4e^{-1/x}}{2e^{-1/x}-1}\right) = 0\left(\dfrac{3}{-1}\right) = 0$

$f(0) = 0$

$\therefore \quad f(x)$ is continuous at $x = 0$.

(b) $$f'(0^+) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{x\left(\dfrac{3e^{1/x}+4}{2-e^{1/x}}\right)}{x}$$

$$= \lim_{x\to 0^+} \frac{3+4e^{-1/x}}{2e^{-1/x}-1} = -3$$

$$f'(0^-) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{3e^{1/x}+4}{2-e^{1/x}} = \frac{4}{2} = 2$$

$f'(0^+) \neq f'(0^-)$

(c) $f'(0^+) = -3$

(d) $f'(0^-) = 2$ exist

4. Given $|f(x)| \leq \sin^2 x$

Clearly $|f(0)| \leq 0 \Rightarrow f(0) = 0$

$$\lim_{x\to 0}|f(x)| = \left|\lim_{x\to 0} f(x)\right| = 0$$

$$|f'(0)| = \left|\lim_{x\to 0}\frac{f(x)-f(0)}{x}\right| = \left|\lim_{x\to 0}\frac{f(x)}{x}\right| \le 0$$

5. $$f(0^-) = \lim_{x\to 0^-}\frac{a\left[1-x\left(x-\frac{x^3}{3!}\ldots\right)\right]+b\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\ldots\right)+5}{x^2} = f(0)$$

$$= \lim_{x\to 0^-}\frac{(a+b+5)-\left(a+\frac{b}{2}\right)x^2+\ldots}{x^2} = 3$$

$$a+b+5=0 \qquad \ldots(1)$$

$$-\left(a+\frac{b}{2}\right)=3$$

$$2a+b=6 \qquad \ldots(2)$$

On solving (1) and (2),

$$\begin{array}{r} a+b+5=0 \\ \underset{-}{2a}+\underset{-}{b}+\underset{-}{6}=0 \\ \hline -a-1=0 \end{array}$$

$$a=-1$$

$$b=-4$$

$$\therefore \qquad a+b=-5$$

$f'(0^+)$ is exist when $c=0$

$$\lim_{x\to 0}(1+dx)^{1/x} = 3$$

$$e^{\lim_{x\to 0}\frac{1}{x}(dx)} = 3$$

$$\Rightarrow \qquad e^d = 3$$

$$d = \ln 3$$

7. (a) $f(x) = \sqrt[3]{x^2|x|}-1-|x|$

But $x^2|x| = |x|^3$

So, $f(x) = |x|-1-|x| = -1$ is every where differentiable.

So, no where non-differentiable.

(b) $\lim\limits_{x\to\infty} [x(\tan^{-1}(x+1) - x\tan^{-1}(x+1))] + [5\tan^{-1}(x+1) - \tan^{-1}(x+1)]$

$$= \lim_{x\to\infty} 4\tan^{-1}(x+1) = 4\left(\frac{\pi}{2}\right) = 2\pi$$

(c) $f(-x) = \sin\left(\ln\left(-x+\sqrt{x^2+1}\right)\right) = \sin\left(\ln\dfrac{1}{x+\sqrt{x^2+1}}\right)$

$= \sin\left(-\ln\left(x+\sqrt{x^2+1}\right)\right) = -\sin\left(\ln\left(x+\sqrt{x^2+1}\right)\right)$

$= -f(x)$

So, $f(x)$ is an odd function.

(d) $f(x) = \dfrac{4-x^2}{4x-x^3}$ is discontinuous at where denominator is zero, $4x - x^3 = 0$

$\Rightarrow \quad x = 0, x = \pm 2$

$\therefore \quad$ a, b, c only correct.

8. $g'(x) = ae^{ax} + f'(x) \quad \Rightarrow \quad g'(0) = a - 5$

$g''(x) = a^2e^{ax} + f''(x)$

$g''(0) = a^2 + 3$

$\Rightarrow \quad a^2 + a - 2 = 0\ ; \quad a = -2, 1$

10. $f(0^+) = f(0) = f(0^-) = 0$

11. $\int f'(x)\,dx = \int f'(-x)\,dx$

$\Rightarrow \quad f(x) + f(-x) = c$

12. $|f(x)| \le x^{4n}$

$\Rightarrow \quad f(0) = 0$

$\lim\limits_{h\to 0}(-h^{4n}) \le \lim\limits_{h\to 0} f(0+h) \le \lim\limits_{h\to 0}(h)^{4n} \qquad \Rightarrow f(0+h) = 0$

$\lim\limits_{h\to 0}(-(-h)^{4n}) \le \lim\limits_{h\to 0} f(0-h) \le \lim\limits_{h\to 0}(-h)^{4n} \Rightarrow f(0-h) = 0$

$\Rightarrow \quad f(x)$ is continuous at $x = 0$.

$$f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = 0 \qquad \left[\lim_{h\to 0}\frac{-h^{4n}}{h} \le \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} \le \lim_{h\to 0}\frac{h^{4n}}{h}\right]$$

$\Rightarrow \quad f(x)$ is differentiable at $x = 0$.

13. $g(x) = 0 \qquad x \in I$

$\quad\;\; = x^2 \qquad x \notin I$

$gof(x) = 0$ for $x \in R$

14. If $f(x)$ is continuous at $x=2$ then $3p+10q=4$

$f(x)$ is differentiable at $x=2$ then $2p+11q=4$

16. $f(x)=x^2 \qquad -2\le x\le 0$

$=x \qquad 0<x<1$

$=x^3 \qquad 1\le x\le 2$

17. $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f(x)\lim_{h\to 0}\frac{\frac{f(x+h)}{f(x)}-1}{h}$$

$$=f(x)\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-1}{h}$$

So, $f'(x)=\frac{f(x)}{x}\cdot f'(1)$

$\ln(f(x))=3\ln x+\ln c$

$f(x)=cx^3$

$f(1)=1$ so $c=1$

$f(x)=x^3$

So, we can check options.

18. $f(x)=(x-1)(x-2)(x+1)(x+2)=(x^2-1)(x^2-4)$

$f'(x)=(x^2-1)2x+(x^2-4)(2x)=2x(2x^2-5)=0$

$x=0,\pm\sqrt{\frac{5}{2}}$

19. If $f(x)$ is continuous at $x=2$ then $3p+10q=4$

$f(x)$ is differentiable at $x=2$ then $2p+11q=4$

20. $y=e^{x\sin x^3}+e^{x\ln(\tan x)}$

$$\frac{dy}{dx}=e^{x\sin(x^3)}[x\cos(x^3)3x^2+\sin(x^3)]+e^{x\ln(\tan x)}\left(\ln(\tan x)+x\frac{1}{\tan x}\sec^2 x\right)$$

$y'=e^{x\sin(x^3)}[3x^3\cos(x^3)+\sin(x^3)]+(\tan x)^x(\ln(\tan x)+2x\operatorname{cosec}2x)$

21. $f(x)=1-(1-x)+(1-x)x^2+(1-x)(1-x^2)x^3+\ldots+(1-x)(1-x^2)\ldots(1-x^{n-1})x^n$

$$=1-(1-x)(1-x^2)(1-x^3)\ldots(1-x^n)=1-\prod_{r=1}^{n}(1-x^r)$$

$$(f(x)-1)=-\prod_{r=1}^{n}(1-x^r)$$

22. $\therefore$ f and g must be continuous.

$$1+a=2+b$$

$$\Rightarrow \quad a=1+b$$

$$3+b=1 \Rightarrow b=-2$$

$$a=-1$$

23. $$f(x) \begin{cases} \rightarrow ax^3+b\,; & 0\le x\le 1 \\ \rightarrow 2\cos \pi x + \tan^{-1}x\,; & 1<x\le 2 \end{cases}$$

is must be continuous and differentiable at $x=1$.

$$\therefore \qquad a+b=-2+\frac{\pi}{4} \qquad \ldots(1) \qquad \text{(continuity)}$$

$$3a=0+\frac{1}{2} \qquad \ldots(2) \qquad \text{(By differentiable)}$$

We get, a and b

24. $$\begin{aligned} f(f(x)) &= 2+x && 0\le x\le 1 \\ &= 2-x && 1<x\le 2 \\ &= 4-x && 2<x\le 3 \end{aligned}$$

25. $\ln(f(x))=\ln(x+1)+\ln(x+2)+\ldots\ldots+\ln(x+100)$

$$\frac{f'(x)}{f(x)}=\frac{1}{x+1}+\frac{1}{x+2}+\ldots\ldots+\frac{1}{x+100}$$

$$\frac{f(x)f''(x)-(f'(x))^2}{(f(x))^2}=-\left(\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\ldots\ldots+\frac{1}{(x+100)^2}\right)$$

if $\quad g(x)=f(x)f''(x)-(f'(x))^2=0$

$$\Rightarrow \quad \frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\ldots\ldots+\frac{1}{(x+100)^2}=0$$

$\Rightarrow$ $g(x)=0$ has no solution.

26. $$\begin{aligned} h(x) &= -1 && x<1 \\ &= |x-2|+a+2-|x| && 1\le x<2 \\ &= |x-2|+a+1-b && x\ge 2 \end{aligned}$$

if $h(x)$ is continuous at $x=1$, then $a=-3$

if $h(x)$ is continuous at $x=2$, then $b=1$

27. $\lim\limits_{x\to 0^-} f(x)=1=\lim\limits_{x\to 0^+} f(x)$

Clearly, $C=1$ and use L' Hospital's rule.

28. Differentiable w.r.t. 'x'

$$2f(x)f'(x)+2y=2f(x+y)f'(x+y)$$

put $x = 0$

$$k + y = f'(y)f(y)$$

integrate on both sides,

$$ky + \frac{y^2}{2} = \frac{f^2(y)}{2} + c \qquad \text{...(1)}$$

put $x = y = 0$ in given equation, we get

$$f^2(0) = 2$$

$f(0) = \sqrt{2}$ as $(f(x) > 0)$

put $y = 0$ in (1)

$$1 + c = 0 \Rightarrow c = -1$$

also put $y = \sqrt{2}$

$$k\sqrt{2} + 1 = \frac{4}{2} - 1$$

$$k\sqrt{2} = 0$$

$$k = 0$$

$$\therefore \qquad \frac{y^2}{2} = \frac{f^2(y)}{2} - 1$$

$$f^2(y) = y^2 + 2$$

$$f(y) = \sqrt{y^2 + 2}$$

$$f(x) = \sqrt{x^2 + 2}$$

Hence, we can answer.

30. $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h \to 0} \frac{f(h)}{h} + x^2 = x^2 + f'(0); f'(x) = x^2 - 1$

32. $f(1^-) = f(1^+) = f(1) = \frac{1}{2}$

$f'(x) = x \qquad 0 \le x < 1$

$= 4x - 3 \qquad 1 \le x \le 2$

$f''(x) = 1 \qquad 0 \le x < 1$

$= 4 \qquad 1 \le x \le 2$

34. $gof(x) = 0$

$fog(x) = 0 \qquad x \in I$

$= [x^2] \qquad x \notin I$

35. $f(g(x)) = x$

$$f'(g(x))g'(x) = 1$$

$$g'(x) = \frac{1}{f'(g(x))}$$

$$g'(e) = \frac{1}{f'(g(e))} = \frac{1}{f'(1)} = \frac{1}{e+1}$$

$$g''(x) = \frac{-1}{(f'(g(x)))^2} f''(g(x)) \cdot g'(x)$$

$$g''(e) = \frac{-1}{(f'(1))^2} f''(1) \cdot g'(e)$$

36. $f(2^+) = \lim\limits_{x \to 2^+} [x-1] = 1$

$$f(2^-) = \lim_{x \to 2^-} \frac{3x - x^2}{2} = 1$$

$$f(3^-) = \lim_{x \to 3^-} [x-1] = 1$$

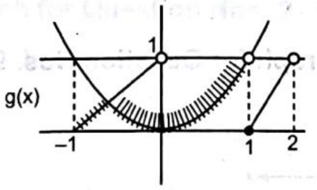
$$f(3^+) = \lim_{x \to 3^+} (x^2 - 8x + 17) = 2$$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

2. $f(x) = \lim\limits_{n \to \infty} n^2 \frac{\tan(\ln(\sec(x/n)))}{\ln(\sec(x/n))} \times \frac{(\ln(\sec(x/n) - 1) + 1)}{\sec(x/n) - 1} \times \frac{\sec(x/n) - 1}{(x/n)^2} \times \left(\frac{x}{n}\right)^2$

$$f(x) = \frac{x^2}{2}$$



Paragraph for Question Nos. 3 to 4

4. $f'(x) = 2x + g'(1)$

$$f''(x) = 2$$

$$f'(1) = 2 - 3 = -1$$

$$g'(1) = 2f(1) + 2 + f'(1)$$

$$\Rightarrow \quad f(1) = -2$$

$f(x) = x^2 - 3x$

$g(x) = -2x^2 + x(2x-3) + 2$

$= -3x + 2$

$f(1) + g(-1) = -2 + (3+2) = 3$

$g''(2) = 2(-2) + 2(2) = 0$

$-2 = 1 + g'(1)$

$g'(1) = -3$

Paragraph for Question Nos. 5 to 6

5. Clearly, 3 is non-repeated root where as 1 is repeats and also $(x-2)^{1/3}$ is not diff. at $x = 2$.

$\therefore$ at 3, 2 is non-diff. and sum is 5.

6. $h(x)$ is continuous.

$x - 1 = x^2 - x - 2$

$x^2 - 2x - 1 = 0$

$(x-1)^2 = 2$

$x = 1 \pm \sqrt{2}$

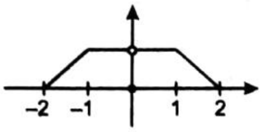
$\tan\frac{3\pi}{8} = 1 + \sqrt{2}, \tan\left(\frac{\pi}{8}\right) = \sqrt{2} - 1$

$\tan\frac{7\pi}{8} = 1 - \sqrt{2}$

$\sqrt{2} - 1$ is not differentiable.

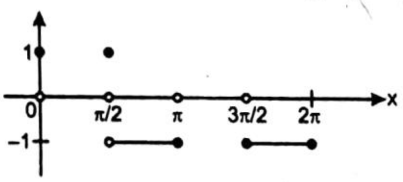
Paragraph for Question Nos. 7 to 8

Sol.



Paragraph for Question Nos. 9 to 10

10.



Paragraph for Question Nos. 11 to 13

11. $f(x) = \left[\begin{array}{cc} 0 & 0 \le x < 1 \\ x & 1 \le x < 2 \\ 2(x-1) & 2 \le x < 3 \\ 3(x-1) & x = 3 \end{array}\right.$

No. of values where $f(x)$ is discontinuous $= 2$

12. $f(x)$ is non-differentiable at $x = 1, 2, 3$.

13. No. of integers in the range of $f(x) = 5$

Paragraph for Question Nos. 14 to 16

Sol. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x)f(h) - f(x)}{h} = f'(0)f(x)$

$\Rightarrow \quad f(x) = e^{2x} \qquad\qquad (f'(0) = 2)$

$g(x) = x^2$

Paragraph for Question Nos. 17 to 18

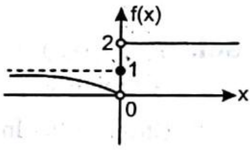
Sol. $g'(x) = \lambda \sec^2 x + (1-\lambda)\cos x - 1 = \dfrac{(1-\cos x)(\lambda - f(x))}{f(x)}$

Paragraph for Question Nos. 19 to 21

Sol. $f(x) = \dfrac{x^2}{x^2+1} \qquad x < 0$

$= 1 \qquad x = 0$

$= 2 \qquad x > 0$



Paragraph for Question Nos. 22 to 24

Sol. $f(x) = g'(1)\sin x + (g''(2) - 1)x$

$\Rightarrow \quad f'(x) = g'(1)\cos x + g''(2) - 1 \;\Rightarrow\; f'\left(\dfrac{\pi}{2}\right) = g''(2) - 1$

$f''(x) = -g'(1)\sin x \qquad \Rightarrow\; f''\left(\dfrac{\pi}{2}\right) = -g'(1)$

$g(x) = x^2 - f'\left(\dfrac{\pi}{2}\right)\cdot x + f''\left(-\dfrac{\pi}{2}\right) \qquad \Rightarrow\; g'(x) = 2x - f'\left(\dfrac{\pi}{2}\right)$

$g''(x) = 2 \qquad \Rightarrow\; g''(2) = 2$

$f(x) = \sin x + x$ and $g(x) = x^2 - x + 1$

Paragraph for Question Nos. 25 to 26

Sol. $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=\lim\limits_{h\to 0}\dfrac{\dfrac{f(x)+f(h)}{1+f(x)f(h)}-f(x)}{h}$

$$f'(x)=\lim_{h\to 0} f(h)\frac{(1-(f(x))^2)}{h}$$

$$f'(x)=\lim_{h\to 0} f'(0)(1-f(x)^2) \qquad (f(0)=0)$$

$$\Rightarrow\ f(x)=\frac{e^{2x}-1}{e^{2x}+1}$$

$$f'(x)\ge 0\ \forall\ x\in R$$

$$\lim_{x\to 0}(f(x))^x=e^{\lim\limits_{x\to 0}\frac{-2x}{(e^{2x}+1)}}=1$$

Paragraph for Question Nos. 27 to 28

Sol. $f(x)=3(x+6)(x+1)(x-2)(x-3)+x^2+1$ then k why!

27. $\lim\limits_{x\to -6}\dfrac{3(x+1)(x-2)(x-3)(x+6)}{x+6}=-\dfrac{6!}{2}$

28. $g(x)=\dfrac{1}{-3(x+6)(x+1)(x-2)(x-3)}$

Paragraph for Question Nos. 29 to 30

Sol. $f(x)=g(x)$

$$x^{\ln x}=e^2x$$

$$(\ln x)^2=2+\ln x$$

$$x=\alpha=\frac{1}{e}\ ,\ \beta=e^2$$

29. $\lim\limits_{x\to e^2}\dfrac{f(x)-c\beta}{g(x)-\beta^2}=\dfrac{f'(x)}{g'(x)}=4$

$$c=e^2$$

30. $h'(\alpha)=\dfrac{g(\alpha)\,f'(\alpha)-g'(\alpha)f(\alpha)}{g^2(\alpha)}=\dfrac{e(-2e^2)-e^2-e}{(e)^2}=-3e$

Exercise-4 : Matching Type Problems

1. (A) Let $I = \int_0^{\pi} \frac{\log(\sin x)}{\cos^2 x} dx$

$$= 2\int_0^{\pi/2} \frac{\log(\sin x)}{\cos^2 x} dx = 2\left[\int_0^{\pi/2} \log(\sin x)\sec^2 x\, dx\right] = 2\left[\log(\sin x)\tan x\Big|_0^{\pi/2} - \int_0^{\pi/2} \frac{\cos x}{\sin x}\tan x\, dx\right]$$

$$= 2(0-0) - 2\int_0^{\pi/2} dx = 0 - 2\left(\frac{\pi}{2}\right) = -\pi = -k$$

$\Rightarrow \qquad k = \pi$

$\therefore \qquad \frac{3k}{\pi} = 3 > 0, 1, 2$

(B) $e^{x+y} + e^{y-x} = 1$

$e^x + e^{-x} = e^{-y}$

$e^x - e^{-x} = e^{-y}(-y')$

$e^x + e^{-x} = e^{-y}(-y'') + e^{-y}(y')^2$

$e^{-y} = e^{-y}(-y'') + e^{-y}(y')^2 \Rightarrow y' - (y')^2 + 1 = 0$

$\therefore \quad k = 1$

(C) Let $f^{-1} = g$

$g\{f(x)\} = x \Rightarrow (g'f(x))f'(x) = 1$

$g'(2\ln 2)\, f'(2) = 1$

$g'2(\ln 2) = \frac{1}{1+\ln 2}$

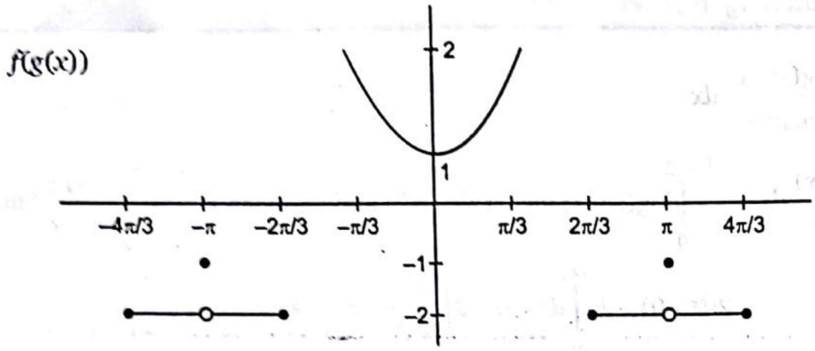
$2(f^{-1})'(\ln 4) = \frac{2}{1+\ln 2} > 0, 1$

(D) $l = \lim_{x\to\infty} (x\ln x)^{\frac{1}{x^2+1}}$

$$\ln l = \lim_{x\to\infty} \frac{\ln x + \ln(\ln x)}{x^2+1} = \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{1}{\ln x}\cdot\frac{1}{x}}{2x} = 0$$

$\ln(l) = 0 \Rightarrow l = 1$

2. $$g(f(x)) \begin{cases} \rightarrow \sec 2, & -2 \le x < -1 \\ \rightarrow \sec 1, & -1 \le x < 0 \\ \rightarrow \sec x, & 0 \le x \le 2 \end{cases}$$



3. (A) $f(1^+) = f(1^-) = -1$

(B) $\int_2^3 ([x]\cdot\{x\} - |x|)\cdot dx = \int_2^3 (2(x-2) - x)\,dx = \left(\frac{x^2}{2} - 4x\right)_2^3 = \frac{-3}{2}$

(C) $[x]\cdot\{x\} = -1 \qquad x \le 0$

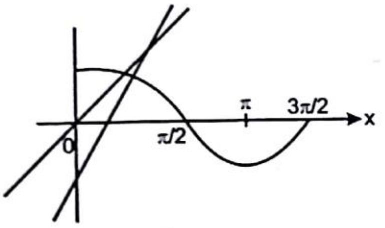
$x = -3 + \frac{1}{3}, -2 + \frac{1}{2}$

(D) $l = \lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} ([x]\{x\} - |x|) = -4$

4. (A) $\lim_{x\to\infty}\left(\frac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+1}{2x-1}} = \frac{1}{2}$

(B) $\lim_{x\to 0}\frac{\log_{\sec x/2}\cos x}{\log_{\sec x}\cos\frac{x}{2}} = \lim_{x\to 0}(\log_{\sec\frac{x}{2}}\cos x)^2 = \lim_{x\to 0}\left(\frac{\ln\cos x}{\ln\sec x/2}\right)^2 = 2$

(C)



(D) $\sin x \ne \frac{1}{3}, \frac{2}{3}, \frac{3}{3}$

5. $f(1^+) = f(1^-) = f(1) \Rightarrow b = 0$

$f(3^-) = f(3^+) = f(3)$

$3 = 9p + 3q + 2 \Rightarrow 3p + q = 0$

$$\begin{aligned} f'(x) &= 2ax - a & x &< 1 \\ &= 1 & 1 \le x &< 3 \\ &= 2px + q & x &> 3 \end{aligned}$$

$f'(3^+) = f'(3^-) = f'(3)$

$6p + q = 1 \Rightarrow p = \frac{1}{3}, q = -1$

$f'(1^+) \neq f'(1^-)$

$a \neq 1$

Exercise-5 : Subjective Type Problems

1. $f(x)$ is discontinuous at $x = 1, |f(x)|$ is diff. every where

$f(1) = -f(1^+) = -f(1^-)$

$\Rightarrow \quad 3 = -(0 + b)$

$\Rightarrow \quad b = -3$

$f'(1^+) = -f'(1^-)$ (as $|f(x)|$ is differentiable every where)

$\therefore \quad 1 = -(2a - a) \Rightarrow a = -1$

Continuous at $x = 3$,

So, $5 = 9p + 3q + 2$

$\Rightarrow \quad 3p + q = 1$ ...(1)

$f'(x)$ is continuous at $x = 3$

So, $f'(3^-) = f'(3^+)$

$\Rightarrow \quad 6p + q = 1$ ...(2)

On solving (1) & (2) we get, $p = 0, q = 1$

So, $|a + b + p + q| = |-1 - 3 + 0 + 1| = 3$

2. $\sin^{-1} y = 8 \sin^{-1} x$

$$\frac{y'}{\sqrt{1-y^2}} = \frac{8}{\sqrt{1-x^2}}$$

$(1-x^2)(y')^2 = 64(1-y^2)$

$(1-x^2)y'' - xy' = -64y$

3. $yy' = 4a$

$(y')^2 + yy'' = 0$

4. $f(x)$ is discontinuous at $x = -\sqrt{3}, -\sqrt{2}, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}$

$\sin \pi x = 0$ at $x = -2, -1, 0, 1, 2, 3$

So, continuous at these points.

5. Let $f'(x) = K$

$\Rightarrow \quad f(x) = Kx + c$

$\Rightarrow\ f(9) - f(-3) = 12K$

Maximum value of $f(9) - f(-3) = 96$

6. $g(x) = \sin x^3 - x^3 + 1 \qquad x \ge 1$

$= \sin x^3 + x^3 - 1 \qquad 0 \le x < 1$

$= -\sin x^3 - x^3 - 1 \qquad -1 \le x < 0$

$= -\sin x^3 + x^3 + 1 \qquad x \le -1$

Function is not differentiable at $x = -1, 1$

7. $F(x) = g(x) \qquad x > 1$

$= \dfrac{f(x) + g(x)}{2} \qquad x = 1$

$= f(x) \qquad -1 < x < 1$

$= \dfrac{f(x) + g(x)}{2} \qquad x = -1$

$= g(x) \qquad x < -1$

If $F(x)$ is continuous at $x = 1$

$F(1^+) = F(1) = F(1^-)$

$b = a + 3$

If $F(x)$ is continuous at $x = -1$

$F(-1^-) = F(-1) = F(-1^+)$

$a + b = 5$

8. $f^{-1}(x) = 2 - x \qquad 2 \le x \le 5$

$= 2 + x \qquad -2 < x < 2$

9. $f(x) + 2f(1 - x) = x^2 + 2$

$f(1 - x) + 2f(x) = (1 - x)^2 + 2 \Rightarrow f(x) = \dfrac{(x - 2)^2}{3}$

10. $g(x) = x(x - 3)(x - 7)$

$f(g(x)) = \operatorname{sgn}(x(x - 3)(x - 7))$

11. $\dfrac{d^2}{dx^2}(\sin^2 x - \sin x + 1) = -4\sin^2 x + \sin x + 2$

12. $f(x) = a\cos(\pi x) + b$

$f'(x) = -a\pi\sin(\pi x)$

$\displaystyle\int_{1/2}^{3/2} f(x)\,dx = -\frac{2a}{\pi} + b = \frac{2}{\pi} + 1 \Rightarrow a = -1,\ b = 1$

13. $\alpha'(x) = f'(x) - 2f'(2x)$; $\beta'(x) = f'(x) - 4f'(4x)$

$\alpha'(1) = f'(1) - 2f'(2) = 5$

$$\alpha'(2) = f'(2) - 2f'(4) = 7$$

$$\beta'(1) = f'(1) - 4f'(4) = \alpha'(1) + 2\alpha'(2) = 5 + (2 \times 7) = 19$$

$$\beta'(1) - 10 = 19 - 10 = 9$$

14. $g(f(x)) = x$

$$g'(f(x))f'(x) = 1$$

$$f(1) = -7/6$$

$$\therefore\ x = 1$$

$$g'\left(-\frac{7}{6}\right)f'(1) = 1$$

$$g'\left(-\frac{7}{6}\right) = \frac{1}{f'(1)}$$

$$f'(x) = -4 \cdot e^{\frac{1-x}{2}}\left(-\frac{1}{2}\right) + x^2 + x + 1$$

$$f'(1) = 2 + 1 + 1 + 1 = 5$$

$$h(x) = ax^{-\frac{5}{4}} + bx^{\frac{1}{4}}$$

$$h'(x) = -\frac{5a}{4}x^{\frac{-9}{4}} + \frac{b}{4}x^{\frac{-3}{4}}$$

$$h'(5) = 0 \Rightarrow -\frac{5a}{4} \cdot 5^{-9/4} + \frac{b}{4} \cdot 5^{-3/4} = 0$$

$$\Rightarrow\ 5a \cdot 5^{-3/2} = b$$

$$\Rightarrow\ \frac{a}{b} = 5^{1/2}$$

$$\left(\frac{a}{b}\right)^2 = 5$$

$$\frac{a^2}{5b^2 g'\left(\frac{-7}{6}\right)} = \frac{5}{5 \times \frac{1}{5}} = 5$$

16. Let $$\lim_{x \to \infty}\left(f(x) + \int_0^x f(t)\,dt\right) = l \qquad \ldots(1)$$

$$\lim_{x \to \infty} \frac{\left(e^x \int_0^x f(t)\,dt\right)}{(e^x)} = l$$

$$\Rightarrow \qquad \lim_{x\to\infty} \frac{e^x \int\limits_0^x f(t)\,dt}{e^x} = l$$

$$\Rightarrow \qquad \lim_{x\to\infty} \int\limits_0^x f(t)\,dt = l \qquad \ldots(2)$$

From (1) and (2) we get, $\lim\limits_{x\to\infty} f(x) = 0$

17. $f(0) = 0,\ f'(0) = 1,\ f''(0) = 1,\ f'''(0) = 2$

$$g(f(x)) = x \qquad \Rightarrow \quad g'(f(x))f'(x) = 1$$

$$\Rightarrow \quad g''(f(x)) = \frac{-f''(x)}{(f'(x))^3}$$

$$\Rightarrow \quad g'''(f(x))f'(x) = -\left[\frac{(f'(x))^3 \cdot f'''(x) - 3(f''(x))^2(f'(x))^2}{(f'(x))^6}\right]$$

Put $x = 0$

$$g'''(0)\,1 = -\left[\frac{1\times 2 - 3\times 1}{1}\right] = 1$$

19. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f\left(x\left(1+\frac{h}{x}\right)\right) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{f(x)}{1+h/x} + \dfrac{f\left(1+\frac{h}{x}\right)}{x} - f(x)}{h} = \lim_{h\to 0} \frac{f(x)\left(-\frac{h}{x}\right)}{h\left(1+\frac{h}{x}\right)} + \frac{f\left(1+\frac{h}{x}\right)}{hx}$$

$$= \frac{-f(x)}{x} + \lim_{h\to 0} \frac{f\left(1+\frac{h}{x}\right) - f(1)}{x^2\left(\frac{h}{x}\right)} \qquad \text{(as } f(1) = 0\text{)}$$

$$f'(x) = \frac{-f(x)}{x} + \frac{f'(1)}{x^2}$$

$$xf'(x) + f(x) = \frac{1}{x}$$

$$\frac{d}{dx}(xf(x)) = \frac{1}{x}$$

$$xf(x) = \int \frac{1}{x}\,dx$$

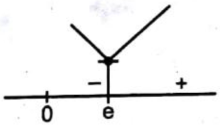
$$xf(x) = \ln x + k$$

Put $x = 1$, we get $k = 0$

$$\therefore \quad f(x) = \frac{\ln x}{x}$$

$$H(x) = \frac{1}{f(x)} = \frac{x}{\ln x}$$

$$H'(x) = \frac{\ln x \cdot 1 - 1}{(\ln x)^2} \qquad H(x) \geq e$$



$$H(e) = e$$

$$\therefore \quad \lim_{x\to e}\left[\frac{1}{f(x)}\right] = 2$$

21. $f'(x) = \tan^{-1}(x^2) + \dfrac{2x^2}{1+x^4} + 4x^3$

$$f''(x) = \frac{2x}{1+x^4} + 2\left(\frac{(1+x^4)\cdot 2x - x^2(4x^3)}{(1+x^4)^2}\right) + 12x^2$$

22. $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = -3\sin\theta\cos\theta = -\dfrac{3}{2}\sin 2\theta$

$$\frac{d^2y}{dx^2} = \frac{-3\cos 2\theta}{\sin\theta}$$

23. Let $8x - 16 = t^2 \Rightarrow \sqrt{\dfrac{t^2+16+8t}{8}} + \sqrt{\dfrac{t^2+16-8t}{8}} = \dfrac{|t+4|+|t-4|}{2\sqrt{2}}$

24. $f(x) = [x] \qquad 0 < x < 1$

$= \{x\} \qquad 1 \leq x < \dfrac{5}{4}$

$= \left|x - \dfrac{3}{2}\right| \qquad \dfrac{5}{4} \leq x < 2$

No. of points where $f(x)$ is non-differentiable are three.

$$x = 1, \frac{5}{4}, \frac{3}{2}$$

* * *

Chapter 4 – Application of Derivatives

4

APPLICATION OF DERIVATIVES

Exercise-1 : Single Choice Problems

1. Maximum value of $f(x) = 3$

Minimum value of $f(x) = -1$

2. $f''(x) = 6x - 6$

$f'(x) = 3x^2 - 6x + 3 \qquad (\because f'(2) = 3)$

$f(x) = x^3 - 3x^2 + 3x - 1 \qquad (\because f(2) = 1)$

5. $V = \frac{4}{3}\pi(10 + T)^3 - \frac{4}{3}\pi(10)^3$

$\frac{dV}{dt} = 4\pi(10 + T)^2 \frac{dT}{dt}$

$\Rightarrow \frac{dT}{dt} = \frac{1}{18\pi} \qquad (\because T = 5\text{ cm})$

6. $g(x) = \frac{(|x| - 1)(|x| - 2)}{(|x| - 3)(|x| - 4)}$

$g(x)$ is an even function so there is an extrema at $x = 0$.

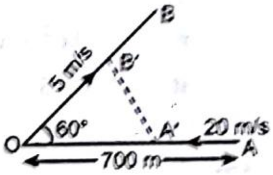
Also number of extrema for $x > 0$ will be equal to number of extrema for $x < 0$

for $x > 0$

$$g(x) = \frac{(x - 1)(x - 2)}{(x - 3)(x - 4)}$$

Number of extrema $= 2$

$\Rightarrow$ Total extrema $= 5$



7. $A'B' = \sqrt{\left(700 - \frac{45}{2}t\right)^2 + \frac{75}{4}t^2}$

$(A'B')_{\min}$ at $t = 30$ sec

8. $f(0^-) \geq f(0) \Rightarrow a \geq 3$

9. $f(x) \begin{cases} 3+k-x, & x \leq k \\ a^2-2+\frac{\sin(x-k)}{x-k}, & x > k \end{cases} \Rightarrow f'(k^+) > f(k), f'(k^-) > f(k)$

So, $\lim\limits_{x \to k^+} (a^2 - 2) + \frac{\sin(x-k)}{(x-k)} = a^2 - 1 > 3$

$a^2 > 4$

$|a| > 2$

10. $\frac{dy}{dx} = 3x^2 - 4x + C_1$

$y = x^3 - 2x^2 + C_1x + C_2$

Also, $\left.\frac{dy}{dx}\right]_{\text{at } x=1} = 0$ and $y]_{\text{at } x=1} = 5$

11. $m_1 = \left.\frac{dy}{dx}\right]_{\text{at }(1,\,2)} = 2a + b$

$m_2 = g'(x) = \left.\frac{dy}{dx}\right]_{\text{at }(-2,\,2)} = 2 \Rightarrow 2a + b = -\frac{1}{2}$

Also, $2 = a + b + \frac{7}{2}$

12. $18y\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{18y}$

$\Rightarrow \frac{a^2}{6b} = 1 \Rightarrow a^2 = 6b$

Also, $9b^2 = a^2$

13. $\frac{dy}{dx} = 3x^2 - 4x + c$

at $x = 1, \frac{dy}{dx} = 0 \quad \Rightarrow c = 1$

$$\frac{dy}{dx}=3x^2-4x+1$$

$$y=x^3-2x^2+x+d$$

at $x=1, y=5 \quad \Rightarrow 5=1-2+1+d$

$\Rightarrow \quad d=5$

14. $A(0,2)$

$$5\alpha^2(3x^2)+10\alpha(2x)+1+2\frac{dy}{dx}=0 \;\Rightarrow\; \frac{dy}{dx}=\frac{-15\alpha^2x^2-20\alpha x-1}{2}$$

$$\Rightarrow \frac{dy}{dx} \text{ at } A=-\frac{1}{2}$$

Equation of normal at A is $y=2x+2$

Let normal meets the curve at B

$$5\alpha^2x^3+10\alpha x^2+x+4x+4-4=0$$

$$5x(\alpha x+1)^2=0$$

$$x=-\frac{1}{\alpha}$$

So, $B\left(\frac{-1}{\alpha},\frac{-2}{\alpha}+2\right)$

$\therefore$ Slope of tangent at $B=\dfrac{-15+20-1}{2}=2$

15. $f(x)=\cos x+\frac{1}{2}\cos 2x-\frac{1}{3}\cos 3x$

$$f'(x)=-\sin x-\sin 2x+\sin 3x=2\sin x(2\cos x+1)(\cos x-1)=0$$

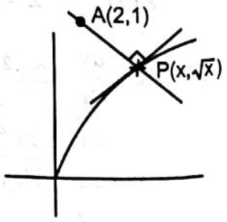
16. Closest distance exist always alone the normal

$$\therefore \quad \frac{1-\sqrt{x}}{2-x}\times\frac{dy}{dx}=-1$$

$$\frac{1-\sqrt{x}}{2-x}\times\frac{1}{2\sqrt{x}}=-1$$

Let $\sqrt{x}=t$

$$x=\frac{2+\sqrt{3}}{2}$$



17. Let $x=2\sin\theta$

$$y=\ln\left(\frac{2+2\cos\theta}{2-2\cos\theta}\right)-2\cos\theta=\ln\left(\frac{2\cos^2\theta/2}{2\sin^2\theta/2}\right)-2\cos\theta$$

$$=2\ln\left(\cot\frac{\theta}{2}\right)-2\cos\theta$$

$$\frac{dy}{d\theta}=\frac{1}{\cot\theta/2}\left(-\text{cosec}^2\frac{\theta}{2}\right)+2\sin\theta$$

$$\frac{dy}{d\theta}=\frac{-2}{\sin\theta}+2\sin\theta=\frac{-2\cos^2\theta}{\sin\theta}$$

$$\frac{dx}{d\theta}=2\cos\theta;\ \frac{dy}{dx}=-\cot\theta$$

$$\left(y-2\ln\left(\cot\frac{\theta}{2}\right)+2\cos\theta\right)=-\cot\theta(x-2\sin\theta)$$

$$\therefore\quad T=\left(0,2\ln\left(\cot\frac{\theta}{2}\right)-2\cos\theta+2\cos\theta\right)$$

$$P=\left(2\sin\theta,2\ln\cot\frac{\theta}{2}-2\cos\theta\right)$$

$$PT^2=(\sqrt{4\sin^2\theta+4\cos^2\theta})=4$$

18. $g'(x)=(2x^2-\ln x)f(x)$

$$f'(x)=\frac{1}{\ln x^3}3x^2-\frac{1}{\ln x^2}2x$$

$$f'(x)=\frac{x^2-x}{\ln x}$$

$$f'(x)=\frac{x(x-1)}{\ln x}>0\ \forall\ x>1;\quad f(x)>f(1)\Rightarrow f(x)>0\ \forall\ x>1$$

For $g(x)$ is increasing

$g'(x)>0\Rightarrow 2x^2-\ln x>0$ as $(f(x)>0)$

Let $H(x)=2x^2-\ln x$

$$H'(x)=4x-\frac{1}{x}=\frac{4x^2-1}{x}>0 \text{ when } x>1$$

$H(x)>H(1)\Rightarrow H(x)>2$

$\therefore\quad g'(x)>0\ \forall\ x\in(1,\infty)$

$\therefore\quad g(x)$ is increasing on $(1,\infty)$.

19. $f'(x)=3x^2+12x+a$

$f'(x)<0$ in $(-3,-1)$

Product of the roots $=\frac{a}{3}=3\Rightarrow a=9$

20. $f(x)=\tan^{-1}\left(\frac{1-x}{1+x}\right)$

$$f'(x)=\frac{1}{1+\left(\frac{1-x}{1+x}\right)^2}\left(\frac{-2}{(1+x)^2}\right)=\frac{2}{2(1+x^2)}=\frac{1}{1+x^2}>0$$

$f'(x)$ is decreasing $\forall\ x \in R$

So, in $[0,1]$ $\quad f(0) = \tan^{-1}(1) = \frac{\pi}{4}$ (max)

$f(1) = 0$ (min)

21. $f'(x) = 3x^2 + 2(a+2)x + 3a$

$\therefore \quad D \le 0$

$a^2 - 5a + 4 \le 0$

$\therefore \quad a \in [1, 4]$

22. $f'(x) = 0$

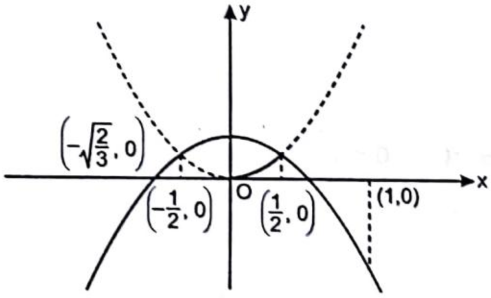
$$\cos^2 x - \sqrt[3]{x} + x^{1/3} - \frac{1}{2} = 0$$

$$\cos^2 x = \frac{1}{2}$$

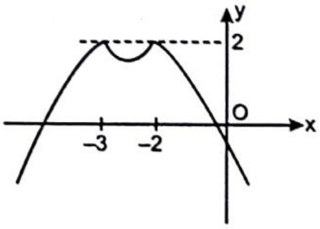
$$\cos x = \pm\frac{1}{\sqrt{2}}$$

$\therefore$ total number is 12.

23.


y
$\left(-\sqrt{\frac{2}{3}}, 0\right)$
$\left(-\frac{1}{2}, 0\right)$
O
$\left(\frac{1}{2}, 0\right)$
(1,0)
x

24.



$b^2 + 1 \ge 2$

25. $f(x)$ is continuous and differentiable in $[-1, 1]$.

26. $\frac{\cos x_1}{x_1} = -\sin x_1 \Rightarrow x_1 = -\cot x_1$

Point $(x_1, \cos x_1)$ always lie on $\frac{1}{y^2} = \frac{1}{x^2} + 1$

27. $x + \dfrac{a}{x^2} > 2 \ \forall\ x \in (0, \infty)$

$f(x) = x^3 - 2x^2 + a > 0 \ \forall\ x \in (0, \infty)$

$f'(x) = 3x^2 - 4x = 3x\left(x - \dfrac{4}{3}\right)$

Minimum value at $x = \dfrac{4}{3}$

$\dfrac{64}{27} - 2\left(\dfrac{16}{9}\right) + a > 0 \ \Rightarrow\ a > \dfrac{32}{27}$

29. $f'(x) = \cos^2 x + \cos x + 2 > 0$

$f(x)_{\min} = f(0) = 0$

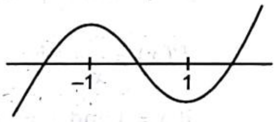
$f(x)_{\max} = f(2\pi) = 5\pi$

31. $f(x) = x^3 - 3x + c = 0$

$f'(x) = 3(x^2 - 1)$

$\Rightarrow\ f(1)f(-1) < 0$

$(c-2)(2+c) < 0$



32. $f'(x) = e^x(x-1)(x-2) < 0$

33. $\dfrac{dy}{dx} = 3ax^2 + 2bx + c = 0$ has one root $\Rightarrow D = b^2 - 3ac = 0 \Rightarrow b^2 = 6$

34. Let $x = \tan\theta$ then $y = \cos^2\theta$

$\left|\dfrac{dy}{dx}\right| = |2\sin\theta\cos^3\theta|$

$\left|\dfrac{dy}{dx}\right|_{\max}$ at $\theta = \dfrac{\pi}{6}$

35. $h(x) = f(x) - g(x) = 2x - 3\sin x + x\cos x$

$h(0) = 0$

$h'(x) = 2 - 2\cos x - x\sin x$

$h'(0) = 0$

$h''(x) = \sin x - x\cos x$

$h''(0) = 0$

$h'''(x) = x\sin x > 0 \ \forall\ x \in \left(0, \dfrac{\pi}{2}\right)$

36. $f(x) = 2\tan^{-1}(g(x)) \qquad |g(x)| \le 1$

$= \pi - 2\tan^{-1} g(x) \qquad g(x) > 1$

$= -\pi - 2\tan^{-1} g(x) \qquad g(x) < -1$

$$f'(x) = \frac{2g'(x)}{1+(g(x))^2} \qquad |g(x)| < 1$$

$$= -\frac{2g'(x)}{1+(g(x))^2} \qquad |g(x)| > 1$$

37. $\lim\limits_{x \to e^a} \left[\frac{7}{3}\left[\frac{\ln(1+7f(x))}{7f(x)}\right] - \frac{1}{3}\left(\frac{\sin f(x)}{f(x)}\right)\right] = 2$

38. If $f(x)$ is strictly decreasing for all x,

$f'(x) = \log_{1/3}(\log_3(\sin x + a)) \leq 0$

$\Rightarrow$ $\sin x + a \geq 3 \ \forall \ x \in R$

$\Rightarrow$ $a \geq 4$

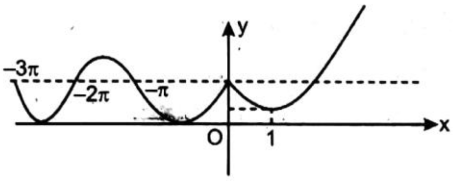
39. $f(x) = a\ln|x| + bx^2 + x$

$$f'(x) = \frac{a}{x} + 2bx + 1 = \frac{2bx^2 + x + a}{x}$$

if $x = 1$ and $x = 3$ are point of extrema.

$\Rightarrow$ $-\frac{1}{2b} = 4$ and $\frac{a}{2b} = 3$

40.



$f(x)$ has local maximum at $x = 0$.

41. $f(x) = \int\limits_1^x (t-a)^{2n}(t-b)^{2m+1}\,dt$

$f'(x) = (x-a)^{2n}(x-b)^{2m+1}$

No sign change of $f'(x)$ about $x = a$.

$f'(x)$ will change sign from negative to positive at $x = b \Rightarrow$ Point of minima.

43. Let point P on the curve $y^2 = x^3$ is $P(t_1^2, t_1^3)$.

Equation of tangent at $P(t_1^2, t_1^3)$ is

$$y - t_1^3 = \frac{3}{2}t_1(x - t_1^2)$$

If this intersect the curve again at $Q(t_2^2, t_2^3)$

$$\Rightarrow \qquad t_2 = -\frac{t_1}{2}$$

$$\frac{\tan\alpha}{\tan\beta} = \frac{(3t_1/2)}{(3t_2/2)} = -2$$

44. $y^2 = \alpha x^3 - \beta$

if $(2, 3)$ is lie on the curve

$$8\alpha - \beta = 9 \qquad \ldots(1)$$

Slope of normal at $(2, 3)$

$$-\frac{1}{4} = -\frac{1}{2\alpha} \quad \Rightarrow \alpha = 2$$

45. Equation of tangent at $(0, 1)$ to the curve $y - 1 = kx$ meet x-axis at $(a, 0)$ then

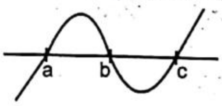
$$-2 \le -\frac{1}{k} \le -1 \quad \Rightarrow k \in \left[\frac{1}{2}, 1\right]$$

46. $f(x) = \int_0^{\sqrt{x}} e^{\frac{-u^2}{x}}\, du = \sqrt{x}\int_0^1 e^{-t^2} dt$

where $t = \frac{u}{\sqrt{x}}$

$\Rightarrow f(x) = K\sqrt{x}, \quad K > 0$

47. $f''(\alpha) = 0 \Rightarrow x = \alpha$ is the point where concavity changes.



48. $f(x) = x^6 - x - 1$

$f'(x) = 6x^5 - 1 > 0 \;\forall\; x \in [1, 2]$

If $f(1) = -1 < 0$ and $f(2) = 2^6 - 3 > 0$ then $f(x)$ has one root in $[1, 2]$.

49. Every line passing from (a, b) is normal to the circle $(x - a)^2 + (y - b)^2 = k$

50. $f'(x) = \cos x(3\sin^2 x - m) = 0$

$\sin^2 x = \frac{m}{3} \quad \Rightarrow 0 < \frac{m}{3} < 1$

$$0 < m < 3$$

51. Let $y = x^{1/x}$

$y' = x^{(1/x)-2}(1 - \ln x)$

$f(x)$ is increasing	$(0, e)$
and $f(x)$ is decreasing	(e, ∞)

52. Let $y = mx$

Point of tangency be (x_1, y_1)

$\Rightarrow \quad mx_1 = x_1^3 + x_1 + 16 \ \& \ m = 3x_1^2 + 1$

$\Rightarrow \quad x_1(3x_1^2 + 1) = x_1^3 + x_1 + 16$

$x_1 = 2$

$m = 13$

53. $y' = 3x^2 - 6x + 6$

$y'' = 6x - 6 = 0$

$x = 1$

$y' = 3$

54. Let $H(x) = \ln(f(x) + f'(x) + \ldots f^n(x)) - x$

$\Rightarrow \quad H(a) = H(b)$

$\Rightarrow \quad H'(c) = 0$ (by L.M.V.T.)

$$\Rightarrow \quad \frac{f'(c) + f''(c) + \ldots + f^{n+1}(c)}{f(c) + f'(c) + \ldots + f^n(c)} - 1 = 0$$

$\Rightarrow \quad f^{n+1}(c) = f(c)$

55. $h(x) = g(x) + x$

$\Rightarrow \quad h'(x) = g'(x) + 1$

$\Rightarrow \quad g'(x) = h'(x) - 1$

$\Rightarrow \quad g''(x) = h''(x)$

$\Rightarrow \quad h''(x) - 3(h'(x) - 1) > 3$

$\Rightarrow \quad h''(x) - 3h'(x) > 0$

$\Rightarrow \quad \frac{d}{dx}(e^{-3x}h'(x)) > 0$

Let $P(x) = e^{-3x}h'(x)$

$\Rightarrow \quad P'(x) > 0$

$\Rightarrow \quad P(x)$ is an increasing function.

$P(0) = h'(0) = 0$

$\Rightarrow \quad P(x) > 0 \ \forall \ x > 0$

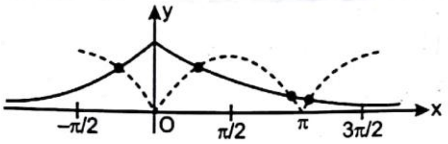
$\Rightarrow \quad h'(x) > 0 \ \forall \ x > 0$

$\Rightarrow \quad h(x)$ is an increasing function $\forall \ x > 0$

56. $\frac{dy}{dx} = -\frac{c}{(x+1)^2} = -1 \Rightarrow (x+1)^2 = c$

Point $(\sqrt{c} - 1, \sqrt{c})$ lie on the line $x + y = 3 \Rightarrow \sqrt{c} = 2$

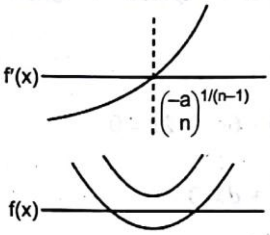
57. $|\sin x| = e^{-x^2}$



60. $x^n + ax + b = 0$

x is even.

$nx^{n-1} + a = f'(x)$



61. $f(b) = \left|\sin x + \dfrac{2}{3+\sin x} + b\right|_{\max} \quad \forall\, x \in R$

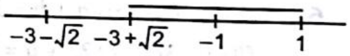
$\sin x = t$

$g(t) = t + \dfrac{2}{3+t} \qquad t \in [-1, 1]$

$g'(t) = 1 - \dfrac{2}{(3+t)^2} > 0$

$(3+t)^2 - 2 > 0$

$(3+t-\sqrt{2})(3+t+\sqrt{2}) > 0$

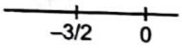


$g(t) = t + \dfrac{2}{3+1}$ increasing $\forall \in [-1, 1]$

$g(t)_{\max} = \dfrac{3}{2}$

$g(t)_{\min} = 0$

$$f(b) = \left|\begin{array}{ll}\dfrac{3}{2} + b & \text{if } b \geq -\dfrac{3}{4} \\ -b & \\ & b < -\dfrac{3}{4}\end{array}\right.$$



min. of $f(b) = -\left(\dfrac{-3}{4}\right) = \dfrac{3}{4}$

62. $y = \dfrac{x}{1+x^2} \Rightarrow \dfrac{dy}{dx} = \dfrac{1-x^2}{(1+x^2)^2}$

63. $f^{-1}(x) = 2\cos^{-1}\left(\dfrac{x}{3}\right)$

$$\frac{d}{dx}f^{-1}(x) = \frac{-2}{\sqrt{1-\dfrac{x^2}{9}}}\left(\frac{1}{3}\right)$$

64. $f(x) = \sin x + \tan x - 2x$

$f'(x) = \cos x + \sec^2 x - 2 = 0$

$\cos^3 x - 2\cos^2 x + 1 = 0 \Rightarrow (\cos x - 1)(\cos^2 x - \cos x - 1) = 0$

$\cos x = 1, \dfrac{1-\sqrt{5}}{2}$

65. $\dfrac{a+2c}{b+3b} + \dfrac{4}{3} = 0 \quad \Leftrightarrow \quad 3a + 4b + 6c + 12b = 0$

$\Leftrightarrow \quad \dfrac{1}{4}a + \dfrac{b}{3} + d = 0$

Consider $f(x) = \dfrac{ax^4}{4} + \dfrac{bx^3}{3} + \dfrac{cx^2}{2} + dx$

then $\quad f(0) = 0 = f(1)$

$\therefore \quad f(x)$ satisfies the conditions of Rolle's theorem in [0, 1].

Hence, $f'(x) = 0$ has at least one solution in (0, 1).

66. $f'(x) = \phi(x)\cdot(x-2)^2$

$\phi(2) > 0 \Rightarrow f'(x) > 0 \Rightarrow f(x)\uparrow$

$\phi(2) < 0 \Rightarrow f'(x) < 0 \Rightarrow f(x)\downarrow$

67. $f(1) = f(3) \Rightarrow a + b - 5 = 3a + b - 27 \Rightarrow a = 11$

$f'(c) = 3c^2 - 12c + a = 0 \Rightarrow b \in R$

70. Let $x = \dfrac{3at}{2^{2/3}}, y = at^{3/2}$

$$\frac{dy}{dx} = \frac{2\left(\dfrac{9a^2t^2}{2^{4/3}}\right)}{9a^2t^{3/2}} = -\cot\alpha \Rightarrow \sqrt{t} = -2^{1/3}\cot\alpha$$

and $P = \cos\alpha\left(\dfrac{3at}{2^{2/3}} - \dfrac{at^{3/2}}{-\cot\alpha}\right)$

$\Rightarrow \quad \dfrac{P}{a} = \cos\alpha\cot^2\alpha$

Exercise-2 : One or More than One Answer is/are Correct

1. Equation of tangent to $y = x^3$

$$y - x_1^3 = 3x_1^2(x - x_1)$$

Equation of tangent to $y = x^{1/3}$ is

$$y - x_2^{1/3} = \frac{1}{3x_2^{2/3}}(x - x_2)$$

If these tangents represent same line

$$\frac{1}{1} = \frac{9x_1^2 x_2^{2/3}}{1} = \frac{-2x_1^3}{\frac{2}{3}x_2^{1/3}} \Rightarrow x_1 = \pm\frac{1}{\sqrt{3}}$$

2. (a) $f'(C_1) = \dfrac{f(4) - f(0)}{4 - 0} = \dfrac{1}{4};\ C_1 \in (0, 4)$

(c) $f'(C_1) = \dfrac{f(8) - f(0)}{8 - 0} = \dfrac{1}{8};\ C_1 \in (0, 8)$

$f(C_2) = f(8) = 1$

(d) Let $g(x) = \displaystyle\int_0^{x^3} f(t)\, dt$

$$\Rightarrow \quad g(0) = 0,\ g(2) = \int_0^8 f(t)\, dt$$

$$g'(\alpha) = 3\alpha^2 f(\alpha^3) = \frac{g(2) - g(0)}{2} \quad \alpha \in (0, 2)$$

and $\quad g'(\beta) = 3\beta^2 f(\beta^3) = \dfrac{g(2) - g(0)}{2} \quad \beta \in (0, 2)$

$$g'(\alpha) + g'(\beta) = g(2) - g(0) = \int_0^8 f(t)\, dt$$

4. $f(x) = 2x^4 + x^4 \sin\dfrac{1}{x} \qquad x \neq 0$

$= 0 \qquad x = 0$

$$f'(x) = 8x^3 + 4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x}$$

5. $-1 \le f''(x) \le 1$

$-x \le f'(x) \le x \qquad (\because f'(0) = 0)$

$-\dfrac{x^2}{2} \le f(x) < \dfrac{x^2}{2} \qquad (\because f(0) = 0)$

6. $f''(x) > 0 \ \forall \ x \in [-3, 4]$

$\Rightarrow \ f'(x)$ is increasing for $x \in [-3, 4]$

7. $f''(x) > 0 \ \forall \ x \in [0, 2]$

$\Rightarrow \ f'(x) \uparrow$

$$f'(C_1) = \frac{f(1) - f(0)}{1 - 0}, C_1 \in (0, 1) \text{ and } f'(C_2) = \frac{f(2) - f(1)}{2 - 1}, C_2 \in (1, 2)$$

$f'(C_1) < f'(C_2) \Rightarrow f(0) + f(2) > 2f(1)$

Similarly applying LMVT between $\left[0, \frac{2}{3}\right]$ and $\left[\frac{2}{3}, 2\right]$

$$\frac{f(2) - f\left(\frac{2}{3}\right)}{\frac{4}{3}} > \frac{f\left(\frac{2}{3}\right) - f(0)}{\frac{2}{3}} \Rightarrow 2f(0) + f(2) > 3f\left(\frac{2}{3}\right)$$

8. Let $g''(x) = a(x - 1)$

$$g'(x) = \frac{ax^2}{2} - ax + b$$

$$g'(-1) = 0 \Rightarrow b = -\frac{3a}{2}$$

$$g(x) = \frac{ax^3}{6} - \frac{ax^2}{2} + bx + c \Rightarrow g(x) = x^3 - 3x^2 - 9x + 5 \qquad (\because g(-1) = 10, g(3) = -22)$$

9. $f(x) = 2x^3 - 3(\lambda + 2)x^2 + 2\lambda x + 5$

$f'(x) = 6x^2 - 6(\lambda + 2)x + 2\lambda = 0$ has two real roots, then

$D > 0 \Rightarrow 3\lambda^2 + 8\lambda + 12 > 0$

$\Rightarrow \ \lambda \in R$

10. $f(x) = 1 + x\ln(x + \sqrt{1 + x^2}) - \sqrt{1 + x^2}$

$f'(x) = \ln(x + \sqrt{1 + x^2})$

$f'(x) \geq 0$ for $\forall \ x \in [0, \infty)$

$f'(x) \leq 0$ for $\forall \ x \in (-\infty, 0]$

11. $f(x, y) = x^m(k - x)^n$

$f'(x, y) = mx^{m-1}(k - x)^n - x^m n \cdot (k - x)^{n-1} = 0$

$$\Rightarrow \ x = \frac{mk}{m + n}$$

$$\text{Maximum value} = \frac{k^{m+n} \cdot m^m \cdot n^n}{(m + n)^{m+n}}$$

12. Let line is tangent at $(3t_1^2, 2t_1^3)$ and normal at $(3t_2^2, 2t_2^3)$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{3t_1^2,\, 2t_1^3} = t_1$$

So, slope of normal at $(3t_2^2, 2t_2^3) = -\frac{1}{t_2}$

$$\Rightarrow t_1 = -\frac{1}{t_2}$$

$$t_2 = -\frac{1}{t_1}$$

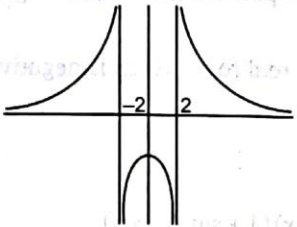
$$\Rightarrow t_1 = \frac{2t_1^3 + \frac{2}{t_1^3}}{3\left(t_1^2 - \frac{1}{t_1^2}\right)}$$

$$\Rightarrow t_1^2(t_1^4 - 3) = 2 \text{ or } t_1 = \pm\sqrt{2}$$

13. $m = \frac{dy}{dx}$

$\frac{|my|}{x+y} = \frac{y}{x}$ then solve it.

14. First draw the graph $f(x) = \frac{1}{x^2 - 4}$



While drawing diff. possibilities of $y = ax^2 + bx + c$

We get possible intersections.

15. $y' = 3x_1^2$

$3x_1^2 = \frac{x_1^3 - 8}{x_1 - 2} \Rightarrow x_1 = -1$ or 2

$y' = 3$ or $y' = 12$

16. Let $f(x) = x + \cos x - a$

$f'(x) = 1 - \sin x \geq 0 \ \forall \ x \in R$

$\Rightarrow$ $f(x)$ is increasing.

$\Rightarrow$ $x + \cos x - a = 0$ for one positive value of x, $a \in (1, \infty)$

17. Let $y = ax + b = f(x) \quad a \neq 0$

$f^{-1}(x) = \dfrac{x-b}{a}$

(1) $m_1 = a, m_2 = \dfrac{1}{a} \qquad \Rightarrow m_1 m_2 = 1$

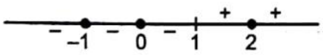
(2) $m_1 = a, m_2 = \dfrac{-1}{a} \qquad \Rightarrow m_1 m_2 = -1$

(3) $m_1 = -a, m_2 = \dfrac{1}{a} \qquad \Rightarrow m_1 m_2 = -1$

(4) $m_1 = -a, m_2 = \dfrac{-1}{a} \qquad \Rightarrow m_1 m_2 = 1$

18. $f'(x) = e^x (x^2 - 1) x^2 (x+1)^{2011} (x-2)^{2012}$

$\qquad = e^x x^2 (x+1)^{2012} (x-1)(x-2)^{2012}$



$x = -1, 0, 2$ are points of inflections and might be more points in (1, 2).

$x = 1$ is point of minima (Answer can be given either d, ad, bd or abd)

19. $f(x) = \sin x + ax + b$

$f'(x) = \cos x + a$

if $a > 1$ then $f(x)$ is increasing.

So, only one real root, which is positive if $b > 0$ and negative if $b > 0$

if $a < -1$

$f(x)$ is decreasing so only one real root, which is negative if $b < 0$.

20. $f''(c) = 0 \quad$ for $\quad c \in (0, 1)$

$f''(x) > 0 \quad$ for $\quad x \in (0, c)$

$f''(x) < 0 \quad$ for $\quad x \in (c, 1)$

21. $f'(x) = 5 \sin x \cos x (\sin x - \cos x)(1 + \sin x \cos x)$

Clearly, $\quad f'(x) > 0 \ \forall \ x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

$\qquad f'(x) < 0 \ \forall \ x \in \left(0, \dfrac{\pi}{4}\right)$

$\qquad f(0) = 0, f\left(\dfrac{\pi}{2}\right) = 0$

By Rolle's theorem $\exists \ c \in \left(0, \dfrac{\pi}{2}\right) \Rightarrow f'(c) = 0$

Clearly, $\quad f(x) \geq f\left(\dfrac{\pi}{4}\right)$

$$f(x) \ge 2\left(\frac{1}{\sqrt{2}}\right)^5 - 1 = 2\left(\frac{1}{4\sqrt{2}}\right) - 1 = \frac{1}{2\sqrt{2}} - 1 \text{ and } f(x) < 0 \ \forall \ x \in \left(0, \frac{\pi}{2}\right)$$

22. $f(x) = x^{2\alpha+1} \ln x \qquad x > 0$

$= 0 \qquad x = 0$

$f(x)$ is not continuous at $\alpha = -\frac{1}{2}, -1$

23. $f'(x) = \frac{\cos x}{x}$

Clearly, $f'(x) > 0 \Rightarrow x \in \left(0, \frac{\pi}{2}\right) \cup \left((4n-1)\frac{\pi}{2}, (4n+1)\frac{\pi}{2}\right) \forall \ n \in N$

and $f'(x) < 0 \Rightarrow x \in \left((4n+1)\frac{\pi}{2}, (4n+3)\frac{\pi}{2}\right) \forall \ n \in N$

$\therefore$ $f(x)$ has a local minima at $x = (4n-1)\frac{\pi}{2} \ \forall \ n \in N$

and $f(x)$ has a local maxima at $x = \frac{\pi}{2}$ and $(4n+1)\frac{\pi}{2} \ \forall \ n \in N$

Also, $f''(x) = \frac{x(-\sin x) - \cos x}{x^2} = 0 \Rightarrow x \tan x + 1 = 0$

$\therefore$ All the points of inflection of $f(x)$ lie on the curve $x \tan x + 1 = 0$

Also, $f'(x) = 0 \Rightarrow x = (2n+1)\frac{\pi}{2} \ \forall \ n \in N$

$\therefore$ Number of values of x in $(0, 10\pi)$ in which $f'(x) = 0$ are 20.

24. $|f(x)| \le 1$

Applying L.M.V.T. in $x \in (0, 1)$

$\Rightarrow$ $|f'(x)| = |f(1) - f(0)|$

$|f(1) - f(0)| \le 2$

$\Rightarrow$ $|f'(x)| \le 2$ for atleast one x in $(0, 1)$

Similarly $|f'(x)| \le 2$ for atleast one x is $(-1, 0)$

$F(x) = (f(x))^2 + (f'(x))^2$

For atleast one x in $(0, 1)$ & $(-1, 0)$

$|f'(x)| \le 2$ & $|f(x)| \le 1$

$\Rightarrow$ $(f'(x))^2 \le 4$ & $(f(x))^2 \le 1$

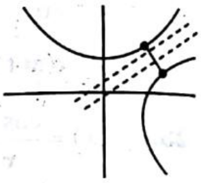
$\Rightarrow$ $(f'(x))^2 + (f(x))^2 \le 5$

$\Rightarrow$ $F(x) \le 5$, for atleast one x in $(-1, 0)$ & $(0, 1)$

25. $f\left(\frac{\pi}{2}\right) > f\left(\frac{\pi^+}{2}\right)$ and $f\left(\frac{\pi}{2}\right) > f\left(\frac{\pi^-}{2}\right)$

Also, absolute maximum occurs at $x = -1$

26. Symmetric about $y = x$



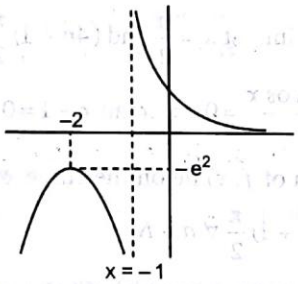
$$\frac{dy}{dx} = 1$$

$$2x = 1 \Rightarrow x = \frac{1}{2}$$

$$\text{Point} = \left(\frac{1}{2}, \frac{5}{4}\right)$$

27. $f'(x) > 0 \quad \Rightarrow \quad \dfrac{-(1+x)\,e^{-x} - e^{-x}}{(1+x)^2} > 0$

for $x < -2$ increasing.



28. Point (1, 2) lies on $y = mx + 5 \quad \Rightarrow m = -3$ ...(1)

Point (1, 2) lies on $x^3y^3 = ax^3 + by^3 \quad \Rightarrow 8b + a = 8$ ...(2)

$3x^2y^3 + x^3 \cdot 3y^2y^1 = 3ax^2 + 3by^2y^1 \Rightarrow a - 12b = -4$ ...(3)

$$\Rightarrow \quad a = \frac{16}{5},\ b = \frac{3}{5}$$

29. $\dfrac{f(x)-1}{f(x)+1} = \dfrac{x^4 + x^2 + 1}{(x^2+x+1)^2} = \dfrac{x^2 - x + 1}{x^2 + x + 1}$

$$\Rightarrow f(x) = \frac{x^2+1}{x}$$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

1. $f(x)=\dfrac{x+1}{x-1}$

$$g(x)=\frac{x(x+1)}{x-1}=x+2+\frac{2}{x-1}$$

$$g'(x)=1-\frac{2}{(x-1)^2}=0$$

$$x=1+\sqrt{2},1-\sqrt{2}$$

$$g''(x)=\frac{4}{(x-1)^3}$$

$$g''(1+\sqrt{2})>0$$

Minimum value of $g(x)$ is $3+2\sqrt{2}$.

2. $g'(x)=\dfrac{1}{2}\Rightarrow 1-\dfrac{2}{(x-1)^2}=\dfrac{1}{2}\Rightarrow x=3,-1$

Paragraph for Question Nos. 3 to 5

3. $g(1)=\displaystyle\int_0^1 f(t)\,dt=\int_0^1(1-t)\,dt=t-\frac{t^2}{2}\Bigg|_0^1=\frac{1}{2}$

4. For $x\in(2,3]$

$$g(x)=\int_0^1 f(t)\,dt+\int_1^2 f(t)\,dt+\int_2^x f(t)\,dt$$

$$g(x)=\frac{1}{2}+\frac{(x-2)^3}{3}$$

at $x=\dfrac{5}{2}$, $g\left(\dfrac{5}{2}\right)=\dfrac{13}{24}$

$$g'(x)=f(x)$$

$$g'\left(\frac{5}{2}\right)=\frac{1}{4}$$

$$y-\frac{13}{24}=\frac{1}{4}\left(x-\frac{5}{2}\right)$$

$$12y=3x-1$$

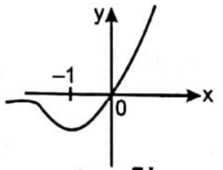
5. Slope of tangent at $P = \frac{1}{4}$

Slope of tangent at $R = \frac{2}{3}$

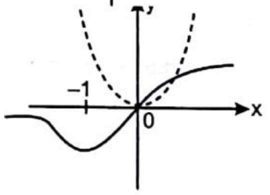
$$\tan\theta = \frac{5}{14}$$

Paragraph for Question Nos. 6 to 8

6.



7.



Paragraph for Question Nos. 9 to 11

9. By putting $x = 1, 2, -1, 0$ we get a, b, c, d clearly other roots product is 1.

10. $P(x) + k = 0$ has 4 distinct real roots.

$P(x) = -k$, where $-k \in (1, 2) \Rightarrow k \in (-2, -1)$

$\therefore$ pull the graph more than 1 and less than 2, now the graph intersect the x-axis in $(-2, -1), (-1, 0), (0, 1), (2, 3)$

$\therefore \quad -2 + (-1) + 0 + 2 = -1$

11. $P(x) = 0$ has two roots.

$P'(x) = 0$ has three root

$P(x) = 0$ has atleast 5 roots.

$(P'(x))^2 + P(x)P''(x) = 0 \Rightarrow (P'(x)P(x))' = 0$ has atleast four roots.

Paragraph for Question Nos. 12 to 14

Sol. The equation of chord AB will be $y - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$

This line passes through $(0, 2x_1x_2)$

$$\therefore \quad 2x_1x_2 - f(x_1) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(-x_1)$$

$$\Rightarrow \quad 2x_1x_2 = \frac{(x_2 - x_1)f(x_1) - x_1 f(x_2) + x_1 f(x_1)}{x_2 - x_1}$$

$$\Rightarrow \quad 2x_1x_2(x_2 - x_1) = x_2 f(x_1) - x_1 f(x_2)$$

$$\Rightarrow \quad \frac{f(x_1)}{x_1} - \frac{f(x_2)}{x_2} = 2(x_2 - x_1) \Rightarrow \frac{f(x_1)}{x_1} + 2x_1 = \frac{f(x_2)}{x_2} + 2x_2 = k$$

$$\therefore \quad \frac{f(x)}{x} + 2x = k$$

$$\therefore \quad f(x) = kx - 2x^2$$

Given that $f(1) = -1$

$\therefore \quad -1 = k - 2 \Rightarrow k = 1$

$\therefore \quad f(x) = x - 2x^2$

12. $\therefore \quad \int_0^{1/2} f(x)\,dx = \left[\frac{x^2}{2} - \frac{2x^3}{3}\right]_0^{1/2} = \frac{1}{8} - \frac{2}{3} \cdot \frac{1}{8} = \frac{1}{8} - \frac{1}{12} = \frac{1}{24}$

13. $f'(x) = 1 - 4x \geq 0 \Rightarrow x \leq \frac{1}{4}$

14. $F(x) = f(x) + x = 2x - 2x^2$

Clearly, $F(0) = F(1) = 0$

$\therefore$ Rolle's theorem is applicable in $[0, 1]$.

Paragraph for Question Nos. 15 to 16

15. $$f(x) = 1 + x\int_0^1 e^y f(y)\,dy + e^x \int_0^1 y\, f(y)\,dy$$

Let $\quad A = \int_0^1 e^y f(y)\,dy, \quad B = \int_0^1 y\, f(y)\,dy$

$$\Rightarrow \quad f(x) = 1 + Ax + Be^x \Rightarrow A = \int_0^1 e^x(1 + Ay + Be^y)\,dy$$

$$B = -\frac{2}{e+1}, A = -\frac{3}{2}$$

$$f'(x) + 3 > 0$$

$$\Rightarrow \quad -\frac{3}{2} - \frac{2e^x}{(e+1)} + 3 > 0 \Rightarrow e^x < \frac{3(e+1)}{4} \Rightarrow \left[\frac{4}{3}e^x\right] = [e+1] = 3$$

16. $Ax_1 + Be^{x_1} = -\frac{3x_1}{2} - 2 \Rightarrow Be^{x_1} = -2$

$f'(x_1) = A + Be^{x_1} = m_1$

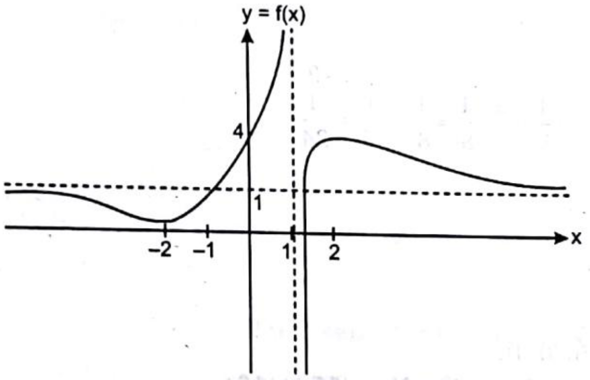
$$m_1 = -\frac{3}{2} - 2 = -\frac{7}{2}$$

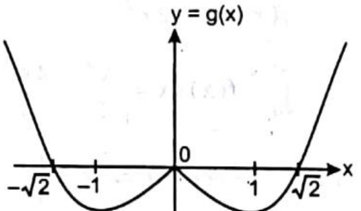
$$m_2 = -\frac{3}{2}$$

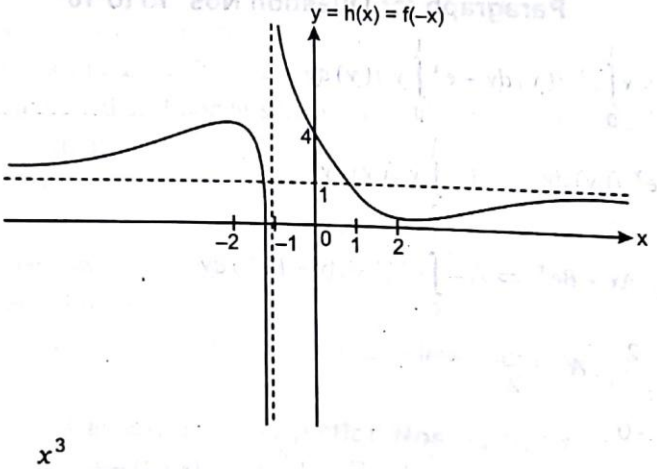
$$\tan\theta = \frac{8}{25}$$

Exercise-4 : Matching Type Problems

3.







4. (A) $y = \dfrac{x^3}{(x-\alpha)(x-\beta)(x-\gamma)}$

$$y = \frac{8}{(2-\alpha)(2-\beta)(2-\gamma)}$$

$$x^3 - 3x^2 + 2x + 4 = (x-\alpha)(x-\beta)(x-\gamma)$$

Put $x = 2$

$8 - 12 + 4 + 4 = (2-\alpha)(2-\beta)(2-\gamma) = 4$

$\therefore \quad y|_{\text{at } x=2} = \frac{8}{4} = 2$

(B) $x^3 + ax + 1 = 0,$ $\quad x^4 + ax + 1 = 0 \qquad \ldots(1)$

$$x^4 + ax^2 + x = 0 \qquad \ldots(2)$$

(2) − (1) gives $\quad ax^2 - ax + x - 1 = 0$

$$ax(x-1) + (x-1) = 0$$

$$(x-1)(ax+1) = 0$$

$$x = 1 \quad \text{or} \quad x = -\frac{1}{a}$$

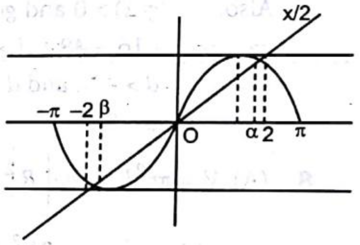
put $x = 1$ in (1) we get, $\quad 1 + a + 1 = 0 \Rightarrow a = -2$

$$|a| = 2$$

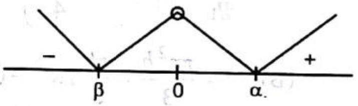
(C) $f(x) = x^2 + 4\cos x + 5$

$f'(x) = 2x - 4\sin x = 2(x - 2\sin x) = 0$

$\sin x = \frac{x}{2}$



$$f'(x) \begin{cases} (>0) & \text{when } x > \alpha \\ <0 & \text{when } 0 < x < \alpha \\ >0 & \text{when } \beta < x < 0 \\ <0 & \text{when } x < \beta \end{cases}$$



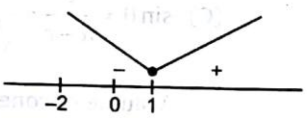
$x = 0$ only a point of maxima

So, number of local maxima is 1.

(D) Let $|x| = t \in [0, 2]$

$\therefore \quad f(x) = 2t^3 + 3t^2 - 12t + 1 = g(t)$

$g'(t) = 6t^2 + 6t - 12t = 6(t^2 + t - 2) = 6(t+2)(t-1)$



$g(1)$ is min. of $f(x)$ *i.e.*, $f_{\min} = 2 + 3 - 12 + 1 = -6$

$g(0) = 1,\ g(2) = 16 + 12 - 24 + 1 = 5$

max. of $f(x)$ is $5 > 4, 3, 2, 0$

6. $f' = \frac{1}{8x} - a + 2x > 0$ since $x > 0$

$1 - 8ax + 16x^2 > 0$

See $D = 0$

$D < 0$

$D > 0$

7. Let $g(x) = ax^3 + bx^2 + cx + d$

$f(x) = \sqrt{g(x)}$

$f(x)$ has local maxima and local maxima at $x = -2$ and $x = 2$.

$\Rightarrow$ $g(x)$ has same local minima and maxima and $x = -2$ and 2.

$\Rightarrow$ $a < 0\,;\, a = -2$

$$f'(x) = \frac{3ax^2 + 2bx + c}{2\sqrt{ax^3 + bx^2 + cx + d}} = 0$$

$f'(-2) = 0$ and $f'(2) = 0$

$\Rightarrow$ $b = 0, c = 24$

Also, $g(2) > 0$ and $g(-2) > 0$

$\Rightarrow$ $-16 + 48 + d > 0$ and $16 - 48 + d > 0$

$d > -32$ and $d > 32$

$\Rightarrow$ $d > 32$

8. (A) $V = \pi r^2 h = \pi h \cdot \left(R^2 - \frac{h^2}{4}\right)$ $\left(\because R^2 = r^2 + \frac{h^2}{4}\right)$

$$\frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right) = 0 \Rightarrow h = \frac{2R}{\sqrt{3}} \text{ and } r = \sqrt{\frac{2}{3}}R$$

(B) $V = \frac{\pi r^2 h}{3} = \frac{\pi h}{3}\{R^2 - (h-R)^2\}$ $\{\because R^2 = r^2 + (h-R)^2\}$

$$\frac{dV}{dh} = \frac{\pi}{3}(4hR - 3h^2) = 0 \quad \Rightarrow h = \frac{4R}{3} \text{ and } r = \frac{2\sqrt{2}R}{3}$$

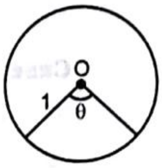
(C) $\sin\theta = \frac{r}{h-r} = \frac{R}{\sqrt{R^2 + h^2}} \quad \Rightarrow R^2 = \frac{h^2 r^2}{h^2 - 2hr}$

Volume of cone $= \frac{\pi}{3}R^2 h = \frac{\pi}{3}\left(\frac{h^2 r^2}{h - 2r}\right)$

$$\frac{dV}{dh} = \frac{\pi}{3}r^2\left(\frac{(h-2r)\,2h - h^2}{(h-2r)^2}\right) = 0 \ \Rightarrow h = 4r$$

(D) $$\frac{\left(\frac{2x}{3} + \frac{2x}{3} + \frac{2x}{3}\right) + \left(\frac{3y}{4} + \frac{3y}{4} + \frac{3y}{4} + \frac{3y}{4}\right)}{7} \geq \left(\frac{8x^3}{27} \cdot \frac{81y^4}{256}\right)^{1/7} \quad \Rightarrow x^3 y^4 \leq \frac{32}{3}$$

Exercise-5 : Subjective Type Problems



1. $$\frac{A_2}{A_1}=\frac{\pi r^2}{\left(\frac{2\pi-\theta}{2\pi}\right)\cdot \pi r^2}=\frac{2\pi}{2\pi-\theta}$$

$$V=\frac{\pi}{3}\left(\frac{\theta}{2\pi}\right)^2\sqrt{1^2-\left(\frac{\theta}{2\pi}\right)^2}$$

$$\frac{dV}{d\theta}=0\ \Rightarrow\ \theta=\sqrt{\frac{8}{3}}\,\pi$$

$$\frac{A_2}{A_1}=\frac{\sqrt{3}}{\sqrt{3}-\sqrt{2}}=3+\sqrt{6}$$

2. $f(x)=x^2\ln x$

$\Rightarrow\ f'(x)=x(1+2\ln x)$

and $f'(x)>0$ for $\in[1,e]$

$\therefore$ $f(x)$ is continuously increasing on $[1, e]$ with the least value zero at $x=1$ and the greatest value e^2 at $x=e$.

3. $f(x)=pxe^{-x}-\dfrac{x^2}{2}+x$

$f'(x)=(1-x)[pe^{-x}+1]\le 0$

$\Rightarrow\ p\le -1$

4. $$f'(x)=\begin{cases}axe^{ax}+e^{ax}; & x\le 0\\ 1+2ax-3x^2; & x>0\end{cases}$$

Clearly, $f'(x)$ is continuous at $x=0$

$$\Rightarrow\ f''(x)=\begin{cases}a^2xe^{ax}+2ae^{ax}; & x\le 0\\ 2a-6x; & x>0\end{cases};$$ $f'(x)$ increasing if $(ax+2)\,ae^{ax}\ge 0$ and $2a-6x\ge 0$

5. $f(x)=x^2-2bx+1$

Case I : $b>1$

$\Rightarrow\quad f(0)-f(1)=4$

$\Rightarrow\quad 1-(2-2b)=4$

$\Rightarrow\quad b=\dfrac{5}{2}$

Case II : $0<b<\dfrac{1}{2}$

$\Rightarrow\quad f(1)-f(b)=4$

$\Rightarrow\quad b=3,-1$ (Not possible)

Case III : $\frac{1}{2} < b < 1$

$\Rightarrow \quad f(0) - f(b) = 4$

$b = \pm 2$ (Not possible)

Case IV : $b < 0$

$\Rightarrow \quad f(1) - f(0) = 4$

$\Rightarrow \quad b = -\frac{3}{2}$

6. $x^2 + 9y^2 = 36 \qquad x^2 + y^2 = 12$

$12 - y^2 + 9y^2 = 36$

$8y^2 = 24 \Rightarrow y^2 = 3 \Rightarrow y = \sqrt{3}, -\sqrt{3}$

when $y = \pm\sqrt{3}$, $x^2 = 12 - 3 = 9$

$x = \pm 3$

$\therefore$ point of intersections are $(\pm 3, \pm\sqrt{3})$

Let one of point of intersect is $(3, \sqrt{3})$

Now, $\frac{2x}{36} + \frac{2y}{4} y' = 0 \Rightarrow y' = \left(-\frac{2x}{36}\right) \times \left(\frac{4}{2y}\right)$

$$(y')_{(3,\sqrt{3})} = -\frac{1}{3\sqrt{3}} = (m_1)$$

$$2x + 2y\,y' = 0 \Rightarrow y' = -\frac{x}{y} \Rightarrow (y')_{(3,\sqrt{3})} = -\frac{3}{\sqrt{3}} = -\sqrt{3} = m_2$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_2 m_1}\right| = \left|\frac{-\frac{1}{3\sqrt{3}} + \sqrt{3}}{1 + \frac{1}{3}}\right| = \left|\frac{8}{4\sqrt{3}}\right| = \left|\frac{2}{\sqrt{3}}\right|$$

$$\tan\theta = \frac{2}{\sqrt{3}} \Rightarrow \theta = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

7. $f'(x) = 2e^{2x} - (\lambda + 1)e^x + 2 \geq 0 \ \forall\ x \in R$

i.e., $2e^{2x} + 2 - (\lambda + 1)e^x$

$\lambda + 1 \leq 2(e^x + e^{-x})$

$\frac{\lambda + 1}{2} \leq (e^x + e^{-x}) \ \forall\ x \in R$

$\Rightarrow \quad \frac{\lambda + 1}{2} \leq (e^x + e^{-x})_{\min} \ \forall\ x \in R$

So, $\frac{\lambda+1}{2} \leq 2$

$\lambda + 1 \leq 4$

$\lambda \leq 3$

$\lambda \in (-\infty, 3]$

$\therefore \quad k = 3$

9. $f(x) = x^2, \qquad g(x) = -\frac{8}{x}$

$\Rightarrow q = p^2 \qquad \Rightarrow \qquad s = -\frac{8}{r}$

Also, $\frac{s-q}{r-p} = 2p \quad \& \quad 2p = \frac{8}{r^2}$

$$pr^2 = 4 \qquad \text{...(1)}$$

$$\frac{s-q}{r-p} = 2p$$

$$\Rightarrow \quad \frac{-\frac{8}{r} - p^2}{r-p} = 2p \quad \Rightarrow \quad -\frac{8}{r} - p^2 = 2pr - 2p^2$$

$$p^2 = \frac{8}{r} + 2pr$$

$$\Rightarrow \quad p^2 r = 16 \qquad \text{...(2)}$$

Multiply (1) & (2),

$\Rightarrow \quad pr = 4$

$\Rightarrow \quad r = 1, \quad p = 4$

10. $f(x) = \begin{bmatrix} |x+2|, & x \geq 0 \\ |x-2|, & x < 0 \end{bmatrix}$

Minimum value of $f(x)$ is 2.

11. $f(x) = \int_0^x [(a-1)(t^2+t+1)^2 - (a+1)(t^4+t^2+1)]\,dt$

$2\int_0^x (t^2+t+1)(at-t^2-1)\,dt$

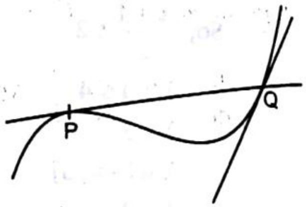
$f'(x) = 2(x^2+x+1)(ax-x^2-1) = 0$

$D < 0 \Rightarrow a^2 - 4 < 0$

12. $f(x) = x^{2013} + e^{2014x}$

$f'(x) = 2013\,x^{2012} + 2014\,e^{2014x} > 0$

$\Rightarrow$ $f(x)$ is increasing function.



14. $P = (x_1, x_1^3 - ax_1)$

$Q = (x_2, x_2^3 - ax_2)$

$y = x^3 - ax$

$\frac{dy}{dx} = 3x^2 - a$

Slope at P = slope of PQ

$$\therefore \qquad (3x_1^2 - a) = \left(\frac{x_2^3 - ax_2 - x_1^3 + ax_1}{x_2 - x_1}\right) \qquad (\because x_1 \neq x_2)$$

$$(x_2 - x_1)(x_2 + 2x_1) = 0$$

$$\Rightarrow \qquad x_2 = -2x_1 \qquad \ldots(1)$$

Slope at P × Slope at $Q = -1$

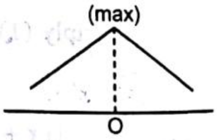
$$(3x_1^2 - a)(3x_2^2 - a) = -1 \qquad \ldots(2)$$

Put (1) in (2),

$$36x_1^4 - 15ax_1^2 + (a^2 + 1) = 0$$

$$D \geq 0 \qquad (\because x_1 \in R)$$

$$9a^2 \geq 16 \Rightarrow a \geq \frac{4}{3}$$



15. $I(t) = \int_{\alpha}^{\beta} (x^2 + 2x - t^2)\,dx = \frac{x^3}{3} + x^2 - t^2 x\Big|_{\alpha}^{\beta}$

$I(t) = \frac{\beta^3 - \alpha^3}{3} + (\beta^2 - \alpha^2) - t^2(\beta - \alpha)$

$I'(t) = -2t(\beta - \alpha) = 0$

$I(t) \leq I(0)$

$$\therefore \quad I(0) = \int_{-2}^{0} (x^2 + 2x)\,dx = -\frac{4}{3} \Rightarrow \frac{p}{q} = |I(0)| = \frac{4}{3}$$

16. $\frac{dy}{dt} = 1 - \frac{3y}{100 - 2t}$ when $t = 0, y = 0$

$\frac{dy}{dt} + \left(\frac{3}{100 - 2t}\right)y = 1$

$y(100 - 2t)^{-3/2} = +(100 - 2t)^{-1/2} + c$

as when $t = 0, y = 0$

$c = -\frac{1}{10}$

$y = (100 - 2t) - \frac{1}{10}(100 - 2t)^{3/2}$

$$\frac{dy}{dt} = -2 - \frac{1}{10}\left(\frac{3}{2}\right)(100-2t)^{1/2}(-2) = 0$$

$$t = \frac{250}{9} = 27 + \frac{7}{9}$$

17. Let $f'(x) = K$

$\Rightarrow \quad f(x) = Kx + c$

$\Rightarrow \quad f(9) - f(-3) = 12K$

Maximum value of $f(9) - f(-3) = 96$

19. Equation of normal at $P\left(\frac{3}{4}y_1^3, y_1\right)$ is $y - y_1 = \frac{-9y_1^2}{4}\left(x - \frac{3}{4}y_1^3\right)$

If it passes from (0, 1) then $27y_1^5 + 16y_1 - 16 = 0$ has only one real root.

20. $e^{-x}\left(\frac{x^2}{2} + x + 1\right) = a$

Let $f(x) = e^{-x}\left(\frac{x^2}{2} + x + 1\right)$

$$f'(x) = e^{-x}\left(-\frac{x^2}{2}\right) < 0$$

21. $f'(x) = a - 2\sin 2x + \cos x - \sin x$

Let $g(x) = -2\sin 2x + \cos x - \sin x$

$$= -2\{(\cos x - \sin x)^2 - 1\} + \cos x - \sin x$$

where $\cos x - \sin x = t$

$$-2t^2 + t + 2 \ \forall \ t \in [-\sqrt{2}, \sqrt{2}]$$

$$-2 - \sqrt{2} \le g(x) \le \frac{17}{8} \Rightarrow a \ge \frac{17}{8}$$

22. Let $\quad x = 6\cos^3\theta, \quad y = 6\sin^3\theta$

$$\frac{dy}{dx} = \frac{6(3\sin^2\theta\cos\theta)}{-6(3\cos^2\theta\sin\theta)} = -\tan\theta$$

Equ. of tangent

$$y - 6\sin^3\theta = -\tan\theta(x - 6\cos^3\theta) \Rightarrow p_1 = 6\sin\theta\cos\theta$$

Equ. of normal

$$y - 6\sin^3\theta = \cot\theta(x - 6\cos^3\theta) \quad \Rightarrow p_2 = 6(\cos^2\theta - \sin^2\theta)$$

$$\sqrt{4p_1^2 + p_2^2} = 6\sqrt{4\sin^2\theta\cos^2\theta + \cos^4\theta + \sin^4\theta - 2\sin^2\theta\cos^2\theta} = 6$$

* * *

Chapter 5 – Indefinite and Definite Integration

5

INDEFINITE AND DEFINITE INTEGRATION

Exercise-1 : Single Choice Problems

1. $\int\left[a^x \ln x + \underbrace{a^x \ln a}_{\text{II}} \cdot \underbrace{x(\ln x - 1)}_{\text{I}}\right]dx$

$$= \int a^x \cdot \ln x\, dx + \left[x(\ln x - 1)\, a^x - \int\left[x \cdot \frac{1}{x} + (\ln x - 1)\right] a^x\right] dx$$

$$= \int a^x \cdot \ln x\, dx + [x(\ln x/e)]\, a^x - \int (\ln x)\, a^x dx$$

2. $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{r=1}^{n} \frac{1}{\sqrt{1+\frac{r}{n}}} = \int\limits_0^1 \frac{dx}{\sqrt{x+1}} = 2(\sqrt{2}-1)$

3. $\int \frac{\sin x}{\sin(x-\alpha)} dx$

Let $x - \alpha = t \Rightarrow dx = dt$

$$\int \frac{\sin(t+\alpha)}{\sin t} dt = t\cos\alpha + \sin\alpha \log \sin t + C = x\cos\alpha + \sin\alpha \log \sin(x-\alpha) + C$$

4. $\int\limits_0^2 \frac{\log(x^2+2)}{(x+2)^2} dx = \left(\frac{-\log(x^2+2)}{x+2}\right)_0^2 + \int\limits_0^2 \frac{2x\,dx}{(x+2)(x^2+2)}$

$$= \frac{\sqrt{2}}{3}\tan^{-1}\sqrt{2} - \frac{5}{12}\log 2 + \frac{1}{12}\log 3$$

5. For $0 < x < 1$

$$1 + x^9 < 1 + x^8 < 1 + x^4 < 1 + x^3$$

6. Let $g(x) = \int\limits_0^x \sqrt{1-(f(s))^2}\, ds$

$$\lim_{t\to x}\left(\frac{g(t)-g(x)}{f(t)-f(x)}\right)=f(x) \qquad \Rightarrow g'(x)=f(x)\,f'(x)=\sqrt{1-(f(x))^2}$$

$$\int\frac{y\,dy}{\sqrt{1-y^2}}=\int dx \qquad \Rightarrow \sqrt{1-y^2}=1-x \qquad \left(\because f\left(\frac{1}{2}\right)=\frac{\sqrt{3}}{2}\right)$$

7. $$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\sqrt{f\left(\frac{r}{n}\right)}=\int_0^1\sqrt{f(x)}\,dx=\frac{2}{\sqrt{3}}\int_0^1\sqrt{1-\cos^6 x-\sin^6 x}\,dx$$

$$=\int_0^1\sin 2x\,dx=\frac{1-\cos 2}{2}$$

8. $$\int_0^1\frac{(x^6-x^3)}{(2x^3+1)^3}\,dx=\frac{1}{2}\int_0^1\frac{2\left(1-\frac{1}{x^3}\right)}{\left(2x+\frac{1}{x^2}\right)^3}\,dx=-\frac{1}{36} \qquad \left(\text{Put } 2x+\frac{1}{x^2}=t\right)$$

9. $$2\int_0^{1/\sqrt{2}}\frac{\sin^{-1}x}{x}\,dx-\int_0^1\frac{\tan^{-1}x}{x}\,dx$$

$$2\int_0^{\pi/4}\frac{\theta\cos\theta}{\sin\theta}\,d\theta-\int_0^{\pi/4}\frac{\theta\sec^2\theta}{\tan\theta}\,d\theta=-\frac{\pi}{4}\ln 2-2\int_0^{\pi/4}\ln\sin\theta\,d\theta+\int_0^{\pi/4}\ln\tan\theta\,d\theta$$

$$=-\int_0^{\pi/4}\ln\sin 2\theta\,d\theta=\frac{\pi}{4}\ln 2$$

10. $$f(x)=x^2+\int_0^x e^{-t}f(x-t)\,dt=x^2-e^{-x}\int_x^0 e^u f(u)\,du \qquad (\text{Let } x-t=u)$$

$$f'(x)=2x+e^{-x}\int_x^0 e^u f(u)\,du+f(x) \quad \Rightarrow f'(x)=x^2+2x$$

$$\Rightarrow f(x)=\frac{x^3}{3}+x^2$$

11. $$f'(x)=f(x)+k_1 \qquad \left(k_1=\int_0^1 f(x)\,dx\right)$$

$$\Rightarrow \quad y=ke^x-k_1$$

If $f(0)=1 \Rightarrow k-k_1=1$

$$k_1=\int_0^1(ke^x-k_1)\,dx \Rightarrow 2k_1=k(e-1) \Rightarrow k=\frac{2}{3-e} \text{ and } k_1=\frac{e-1}{3-e}$$

12. $I_1 = \int\limits_{\sin^2 x}^{1+\cos^2 x} tf(t(2-t))\,dt = 2\int\limits_{\sin^2 x}^{1+\cos^2 x} f(t(2-t))\,dt - \int\limits_{\sin^2 x}^{1+\cos^2 x} tf(t(2-t))\,dt$

$I_1 = 2I_2 - I_1 \Rightarrow I_1 = I_2$

13. $\int \frac{5\sin x\,dx}{\sin x - 2\cos x} = \int dx + 2\int \frac{\cos x + 2\sin x}{\sin x - 2\cos x}dx = x + 2\ln|\sin x - 2\cos x| + C$

14. $\int \frac{(2+\sqrt{x})\,dx}{(x+1+\sqrt{x})^2} = \int \frac{\frac{2}{x^2} + \frac{1}{x^{3/2}}}{\left(\frac{1}{x} + \frac{1}{\sqrt{x}} + 1\right)^2}dx$

Let $\frac{1}{x} + \frac{1}{\sqrt{x}} + 1 = t \Rightarrow \left(-\frac{1}{x^2} - \frac{1}{2x^{3/2}}\right)dx = dt$

15. $\int \frac{\left(\sqrt[3]{x+\sqrt{2-x^2}}\right)\left(\sqrt[6]{1-x\sqrt{2-x^2}}\right)dx}{\sqrt[3]{1-x^2}} = \int \frac{\sqrt[3]{x+\sqrt{(2-x^2)}}\sqrt[3]{\left(\frac{\sqrt{2-x^2}-x}{2}\right)}}{\sqrt[3]{1-x^2}}dx$

$= 2^{1/6}\int dx = 2^{1/6}x + C$

16. $\int \frac{dx}{\sqrt{1-\tan^2 x}}$

$\int \frac{\cos x\,dx}{\sqrt{1-2\sin^2 x}} = \frac{1}{\sqrt{2}}\sin^{-1}(\sqrt{2}\sin x) + C$

17. $I = \int \frac{dx}{x^{5/6}\cdot(x+1)^{7/6}}$

$\frac{x}{x+1} = t \qquad \frac{dx}{(x+1)^2} = dt$

$I = \int \frac{(x+1)^2\,dt}{[t(x+1)]^{5/6}(x+1)^{7/6}} = \int t^{-5/6}dt = 6t^{1/6} + C$

18. $I_n = \int \sin^n x\,dx = \int \sin^{n-2} x(1-\cos^2 x)\,dx$

$I_n = I_{n-2} - \int \underbrace{\sin^{n-2} x\cdot\cos x}_{\text{II}} \cdot \underbrace{\cos x}_{\text{I}}\,dx$

$I_n = I_{n-2} - \left(\frac{\cos x\cdot\sin^{n-1} x}{n-1} - \int -\sin x\cdot\frac{\sin^{n-1} x}{n-1}dx\right)$

$I_n = I_{n-2} - \frac{\cos x\cdot\sin^{n-1} x}{n-1} - \frac{1}{n-1}I_n$

$nI_n - (n-1)I_{n-2} = -\cos x\cdot\sin^{n-1} x$

19. $\int x^2 \frac{1}{(a+bx)^2}\,dx$ Let $a+bx=t$ then $dx=\frac{dt}{b}$

$$\therefore \int x^2\frac{1}{(a+bx)^2}\,dx=\int\left(\frac{t-a}{b}\right)^2\cdot\frac{1}{t^2}\cdot\frac{dt}{b}=\frac{1}{b^3}\int\left(\frac{t^2-2at+a^2}{t^2}\right)dt$$

$$=\frac{1}{b^3}\int\left(1-\frac{2a}{t}+\frac{a^2}{t^2}\right)dt=\frac{1}{b^3}\left[t-2a\ln|t|-\frac{a^2}{t}\right]+C$$

$$=\frac{1}{b^3}\left[a+bx-2a\ln|a+bx|-\frac{a^2}{a+bx}\right]+C$$

20. $\int\frac{8x^{43}+13x^{38}}{(x^{13}+x^5+1)^4}\,dx$

$$\int\frac{8x^{-9}+13x^{-14}}{(1+x^{-8}+x^{-13})^4}\,dx$$

Let $1+x^{-8}+x^{-13}=t$

$(-8x^{-9}-13x^{-14})\,dx=dt$

$$\therefore \int-\frac{dt}{t^4}=+\frac{1}{3t^3}+C=\frac{1}{3(1+x^{-8}+x^{-13})^3}+C=\frac{x^{39}}{3(x^{13}+x^5+1)^{13}}+C$$

21. $\int\left(\frac{(\cos 6x+\cos 4x)+5(\cos 4x+\cos 2x)+10(\cos 2x+1)}{10\cos^2 x+5\cos x\cos 3x+\cos x\cos 5x}\right)dx$

$$=\int\frac{2\cos 5x\cos x+5(2\cos 3x\cos x)+10(2\cos^2 x)}{10\cos^2 x+5\cos x\cos 3x+\cos x\cos 5x}\,dx$$

$$=\int 2\,dx=2x+C$$

$f(x)=2x \Rightarrow f(10)=20$

22. $\int(1+x-x^{-1})\,e^{x+x^{-1}}\,dx=\int e^{x+x^{-1}}\cdot 1\,dx+\int(x+x^{-1})\,e^{x+x^{-1}}\,dx$

$$=e^{x+x^{-1}}\cdot x-\int e^{x+x^{-1}}\left(1-\frac{1}{x^2}\right)x\,dx+\int(x-x^{-1})\,e^{x+x^{-1}}\,dx+C$$

$$=xe^{x+x^{-1}}+C$$

23. $\int e^x\left[\frac{2\tan x}{1+\tan x}+\frac{1}{\left(\frac{1-\cos(\pi/2+2x)}{2}\right)}\right]dx=\int e^x\left[\frac{2\tan x}{1+\tan x}+\frac{2}{(1+\sin 2x)}\right]dx$

$$=2\int e^x\left[\frac{\sin x}{\sin x+\cos x}+\frac{1}{(\sin x+\cos x)^2}\right]dt$$

Let $f(x)=\frac{\sin x}{\sin x+\cos x}$, $f'(x)=\frac{1}{(\sin x+\cos x)^2}$

$$=2\cdot e^x\left(\frac{\sin x}{\sin x+\cos x}\right)+C$$

$g\left(\frac{5\pi}{4}\right)=1$

24. $\frac{d}{dx}(x\sin x+\cos x)=x\cos x$ $\quad f'(x)=x\cos x$

Let $f(x)=x\sin x+\cos x$ $\quad f''(x)=-x\sin x+\cos x$

$$\int e^{f(x)}\left(xf'(x)+\frac{f''(x)}{(f'(x)^2)}\right)dx=\int xe^{f(x)}f'(x)\,dx+\int e^{f(x)}\cdot\frac{f''(x)}{(f'(x)^2)}dx$$

$$=xe^{f(x)}-\int e^{f(x)}dx+e^{f(x)}\frac{-1}{f'(x)}-\int e^{f(x)}\cdot f'(x)\left(\frac{-1}{f'(x)}\right)dx$$

$$=xe^{f(x)}-\frac{e^{f(x)}}{f'(x)}+C=e^{f(x)}\left(x-\frac{1}{f'(x)}\right)+C$$

$$=e^{x\sin x+\cos x}\left(x-\frac{1}{x\cos x}\right)+C$$

25. $\int_0^1\left(\sqrt{x}+\frac{1}{\sqrt{x}+\sqrt{1+x}}\right)dx$

$$\int_0^1(\sqrt{x}+(\sqrt{1+x}-\sqrt{x})\,dx=\int_0^1(\sqrt{1+x}\,dx=\frac{2}{3}(1+x)^{3/2}\Big|_0^1=\frac{2}{3}(2^{3/2}-1)$$

26. $\int x^{x^2}x(2\ln x+1)\,dx$

$x^{x^2}=t$

$x^2\ln x=\ln t$

$\left(x^2\frac{1}{x}+(\ln x)2x\right)dx=\frac{1}{t}dt$

$\therefore$ $\int t\cdot\frac{dt}{t}=\int dt=t+C=x^{x^2}+C=(x^x)^x+C$

27. $=\int\sec^{2010}x\,\text{cosec}^2x\,dx-\int 2010\sec^{2010}x\,dx$

$$=\int\sec^{2010}x(-\cot x)-\int 2010\sec^{2010}x\cdot\tan x\cdot(-\cot x)-\int 2010\sec^{2010}x\,dx$$

$$=-\frac{\cot x}{(\cos x)^{2010}}+2010\int\sec^{2010}x\,dx-2010\int\sec^{2010}x\,dx+C=\frac{-\cot x}{(\cos x)^{2010}}+C$$

$\therefore$ $\frac{f(x)}{g(x)}=\frac{1}{\sin x}=\{x\}$ no solution.

28. Let $x^x \ln x = t$

$$\Rightarrow \left(x^x \ln x(1+\ln x)+\frac{x^x}{x}\right)dx = dt$$

$$\Rightarrow x^x\left(\ln x+(\ln x)^2+\frac{1}{x}\right)dx = dt$$

$$\int dt = t + C = x^x \ln x + C$$

29. $$I=\int\frac{\left(\frac{1}{x^3}-\frac{1}{x^5}\right)dx}{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}$$

Let $2-\frac{2}{x^2}+\frac{1}{x^4}=t \Rightarrow I=\frac{1}{4}\int\frac{dt}{\sqrt{t}}$

30. Put $\ln x = t$

$$I=\int e^t\left(\frac{t-1}{t^2+1}\right)^2 dt=\int e^t\left(\frac{1}{t^2+1}-\frac{2t}{(t^2+1)^2}\right)dt$$

31. $$I=\int\frac{dx}{(x-1)^{3/4}(x+2)^{5/4}}=\int\frac{dx}{\left(\frac{x-1}{x+2}\right)^{3/4}(x+2)^2}$$

Let $\frac{x-1}{x+2}=t \Rightarrow dt=\frac{3dx}{(x+2)^2}$

32. $$\int\frac{2-(1+x^7)}{x(1+x^7)}dx=-\int\frac{dx}{x}+\frac{2}{7}\int\frac{7x^6}{x^7(1+x^7)}dx=-\ln|x|+\frac{2}{7}\ln|1+x^7|+C$$

33. $$I=\int\frac{(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)}{1-2\sin^2 x\cos^2 x}dx=\int(\sin^2 x-\cos^2 x)\,dx=-\int\cos 2x\,dx$$

34. $$I=2^{1/3}\int\frac{(\tan x)^{1/3}d((\tan x)^{1/3})}{(\tan x)^{2/3}+1}$$

Let $(\tan x)^{1/3}=t \Rightarrow d((\tan x)^{1/3})=dt$

$$I=\frac{2^{1/3}}{2}\int\frac{2t}{t^2+1}dt$$

35. $$\int\frac{(2012)^x}{\sqrt{1-(2012)^{2x}}}\cdot(2012)^{\sin^{-1}(2012)^x}dx$$

Let $\sin^{-1}(2012)^x=t \Rightarrow \frac{1}{\ln 2012}\int(2012)^t\,dt=\frac{(2012)^{\sin^{-1}(2012)^x}}{\ln^2(2012)}+C$

36. Let $x+1=t^2 \Rightarrow dx=2t\,dt$

$$2\int\frac{(t^2+1)\,dt}{t^4+t^2+1}=2\int\frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+3}$$

37. $\int\left(\frac{f(x)g'(x)-f'(x)g(x)}{f(x)g(x)}\right)\ln\left(\frac{g(x)}{f(x)}\right)dx$

Let $\frac{g(x)}{f(x)}=t \Rightarrow \frac{f(x)g'(x)-g(x)f'(x)}{(f(x))^2}dx=dt;\quad \int\frac{\ln t}{t}dt=\frac{(\ln t)^2}{2}+C$

38. $\int\left(\int e^x\left(\ln x+\frac{2}{x}-\frac{1}{x^2}\right)dx\right)dx$

$$\int\left(\int e^x\left(\ln x+\frac{1}{x}\right)dx+\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx\right)dx=\int\left(e^x\left(\ln x+\frac{1}{x}\right)+C_1\right)dx=e^x\ln x+C_1x+C_2$$

39. $f(x)=\pi^2\left(\left|\frac{-t\cos(x+\pi t)}{\pi}\right|_0^1+\int_0^1\frac{1\cdot\cos(x+\pi t)}{\pi}dt\right)=\pi\cos x-2\sin x$

40. $\frac{2}{x}\le\sqrt{5}\Rightarrow x\in\left[\frac{2}{\sqrt{5}},1\right)$

$\therefore\quad \int_0^1 f(x)\,dx=\int_0^{2/\sqrt{5}}f(x)\,dx+\int_{2/\sqrt{5}}^1 f(x)\,dx\le\sqrt{5}\left(\frac{2}{\sqrt{5}}-0\right)+\int_{2/\sqrt{5}}^1\frac{2}{x}dx$

$\therefore\quad \int_0^1 f(x)\,dx\le 2+2[\ln x]_{2/\sqrt{5}}^1$

$\therefore\quad a=2+2\ln\left[\frac{\sqrt{5}}{2}\right]$

42. $f(0)=0,\ f(2\pi)=2\pi$

$\therefore\quad \int_0^{2\pi}f(x)\,dx+\int_0^{2\pi}f^{-1}(x)\,dx=\int_0^{2\pi}2\pi\,dx=4\pi^2$

$\Rightarrow\quad \left[\frac{x^2}{2}-\cos x\right]_0^{2\pi}+I=4\pi^2\ \Rightarrow\ I=\int_0^{2\pi}f^{-1}(x)\,dx=2\pi^2$

43. $=2\left[2\int_0^1 e^{-x^4}dx-\int_0^1 8x^4e^{-x^4}dx\right]=2\left[2\left[(xe^{-x^4})_0^1+\int_0^1 4x^4e^{-x^4}dx\right]-\int_0^1 8x^4e^{-x^4}dx\right]$

$=\frac{4}{e}$

46. Put $y-2=z$

$$I=\int_{-2}^{2}\frac{z^2+1}{2z^2+3}\sin(z)\,dz=0$$

47. $\displaystyle\int_{1}^{4}\frac{3}{x}e^{\sin x^3}\,dx$

Let $x^3=t \Rightarrow 3x^2dx=dt$

$$\int_{1}^{64}\frac{e^{\sin t}}{t}\,dt=F(64)-F(1)$$

51. $\displaystyle\lim_{x\to\infty}x\int_{0}^{x}e^{t^2-x^2}\,dt=\lim_{x\to\infty}\frac{x\cdot\int_{0}^{x}e^{t^2}dt}{e^{x^2}}$

Apply L' Hospital's rule,

$$\lim_{x\to\infty}\frac{x\cdot(e^{x^2})+\int_{0}^{x}e^{t^2}dt\cdot 1}{e^{x^2}\cdot 2x}=\lim_{x\to\infty}\left(\frac{1}{2}+\frac{\int_{0}^{x}e^{t^2}dt}{2xe^{x^2}}\right)=\frac{1}{2}$$

52. $\displaystyle L=\sum_{r=1}^{n}\frac{2\cdot r+n}{r^2+n\cdot r+n^2}=\int_{0}^{1}\frac{(2x+1)\,dx}{x^2+x+1}=\ln(x^2+x+1)\Big|_0^1$

$L=\ln 3$

53. Let $\sqrt[3]{x^2+2x}=y=f(x)$

$x=-1+(y^3+1)^{1/2}$

$$I=\int_{0}^{2}(f^{-1}(x)+f(x)+1)\,dx$$

Consider $\displaystyle\int_{0}^{2}f^{-1}(x)=\int_{0}^{2}tf'(t)\,dt$ Let $f^{-1}(x)=t;x=f(t);dx=f'(t)\,dt$

$$=tf(t)\Big|_0^2-\int_{0}^{2}dx=6$$

54. Put $x=2\tan\theta$ then $\displaystyle I=\int_{0}^{\pi/2}\left(\frac{\ln 2+\ln\tan\theta}{4\sec^2\theta}\right)2\sec^2\theta\,d\theta$ then solve it.

55. Put $x-5=t$

$x=0, t=-5$

$x=10, t=5$

$$\int_{-5}^{5}(t+t^2+t^3)\,dt=\left.\frac{t^3}{3}\right|_{-5}^{5}=\frac{250}{3}$$

56. Let $$I=\int_0^\infty \frac{dx}{(1+x^9)(1+x^2)} \qquad ...(1)$$

Put $x=\frac{1}{t} \Rightarrow dx=-\frac{1}{t^2}dt$

$$I=\int_\infty^0 \frac{-\frac{dt}{t^2}}{\left(\frac{t^9+1}{t^9}\right)\left(\frac{1+t^2}{t^2}\right)}=\int_0^\infty \frac{t^9\,dt}{(t^9+1)(1+t^2)} \qquad ...(2)$$

On adding (1) & (2),

$$2I=\int_0^\infty \frac{dt}{(1+t^2)}=\tan^{-1}t\Big|_0^\infty$$

$$2I=\frac{\pi}{2} \Rightarrow I=\frac{\pi}{4}$$

57. $$I=\int_0^{\pi/2}\left(\frac{1+\sin 3x}{1+2\sin x}\right)dx=\int_0^{\pi/2}\frac{1+3\sin x-4\sin^3 x}{1+2\sin x}dx$$

$$=\int_0^{\pi/2}\frac{(1+2\sin x)(-2\sin^2 x+\sin x+1)}{(1+2\sin x)}=-2\left(\frac{1}{2}\frac{\pi}{2}\right)+1+\frac{\pi}{2}=1$$

58. $$\lim_{x\to\infty}\frac{(\tan^{-1}x)^2}{\frac{1}{2\sqrt{x^2+1}}2x}$$

$$\lim_{x\to\infty}(\tan^{-1}x)^2\,\frac{\sqrt{1+x^2}}{x}=\frac{\pi^2}{4}$$

59. Let $t=\prod_{r=1}^{2013}(x^2+r^2)$

$$\ln t=\sum_{r=1}^{2013}\ln(x^2+r^2)$$

$$\frac{1}{t}dt=\sum_{r=1}^{2013}\frac{2x}{x^2+r^2}dx \Rightarrow dt=2\left(\sum_{r=1}^{2013}\frac{x}{x^2+r^2}\right)t\,dx$$

$$\therefore \int \frac{dt}{2} = \frac{t}{2} = \frac{1}{2}\left(\prod_{r=1}^{2013}(x^2+r^2)\right)_0^1 = \frac{1}{2}\left(\prod_{r=1}^{2013}(1+r^2) - \prod_{r=1}^{2013} r^2\right)$$

60. $f'(x) = 2 - \frac{1}{1+x^2} - \frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right) = 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$

$$= \frac{2(1+x^2)-1-\sqrt{1+x^2}}{1+x^2} = \frac{\left((1+x^2)-\sqrt{1+x^2}\right)+x^2}{1+x^2} > 0 \ \forall\ x \in R$$

61. $I = \int_0^{\pi/2} \frac{\sin x \cos x\, dx}{1+\sin x+\cos x} = \int_0^{\pi/2} \sin\frac{x}{2}\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right) dx$

$$= \frac{1}{2}\int_0^{\pi/2} [\sin x + \cos x - 1]\, dx = 1 - \frac{\pi}{4}$$

62. $I = \int_3^7 \frac{\cos x^2 dx}{\cos x^2 + \cos(10-x)^2}$

$I = \int_3^7 \frac{\cos(10-x)^2 dx}{\cos(10-x)^2 + \cos x^2}$

$\Rightarrow \ 2I = 4 \ \Rightarrow I = 2$

63. $-\int_{e^{-1}}^{1} \frac{\ln x}{x} dx + \int_1^{e^2} \frac{\ln x}{x} dx = -\left[\frac{(\ln x)^2}{2}\right]_{e^{-1}}^1 + \left[\frac{(\ln x)^2}{2}\right]_1^{e^2} = 2 + \frac{1}{2} = \frac{5}{2}$

64. $\lim\limits_{x\to\frac{\pi}{4}} \frac{\int_2^{\mathrm{cosec}^2 x} tg(t)\, dt}{x^2 - \frac{\pi^2}{16}} = \lim\limits_{x\to\frac{\pi}{4}} \frac{\mathrm{cosec}^2 x\, g(\mathrm{cosec}^2 x)\, 2\mathrm{cosec}\, x(-\mathrm{cosec}\, x)\cot x}{2x} = \frac{-16}{\pi} g(2)$

65. $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \frac{n-k}{n^2} \cos\frac{4k}{n}$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\left(1-\frac{k}{n}\right)}{n} \cos\left(\frac{4k}{n}\right) = \int_0^1 (1-x)\cos 4x\, dx$$

$$= (1-x)\frac{\sin 4x}{4}\Big|_0^1 + \int_0^1 \frac{\sin 4x}{4} dx = -\frac{1}{16}\cos 4x\Big|_0^1$$

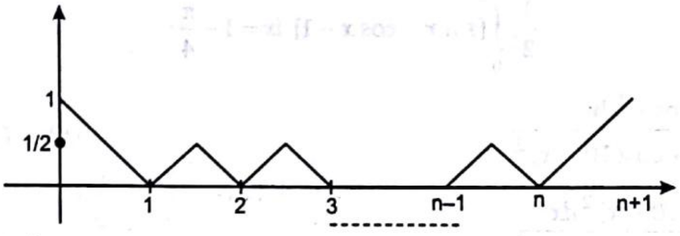
$$= -\frac{1}{16}(\cos 4 - 1) = \frac{1}{16}(1-\cos 4)$$

66. $$\lim_{n\to\infty}\left[\int_0^{1/n}\sin\frac{\pi}{2n}\,dx+\int_{1/n}^{2/n}\sin\frac{2\pi}{2n}\,dx+\ldots\ldots+\int_{1-1/n}^{1}\sin\frac{n\pi}{2n}\,dx\right]$$

$$\lim_{n\to\infty}\frac{1}{n}\left[\sin\frac{\pi}{2n}+\sin\frac{2\pi}{2n}+\ldots\ldots+\sin\frac{n\pi}{2n}\right]$$

$$\lim_{n\to\infty}\frac{\sin\left(\frac{n\pi}{4n}\right)}{n\sin\frac{\pi}{4n}}\sin\left(\frac{(n+1)\pi}{4n}\right)=\frac{2}{\pi}$$

67. $$\int_0^{n+1}\min.\{|x-1|,|x-2|,|x-3|,\ldots\ldots|x-n|\}\,dx=\frac{1}{2}(1)+\frac{1}{2}\times\frac{1}{2}\times(n-1)+\frac{1}{2}\times(1)=\frac{n+3}{4}$$



68. $$S_k=\frac{1}{2}k\sin\left(\frac{k\pi}{2n}\right)$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{2}k\sin\frac{k\pi}{2n}=\int_0^1\frac{1}{2}x\sin\left(\frac{\pi x}{2}\right)dx=\frac{2}{\pi^2}$$

71. $$f(x)=\int_0^{g(x)}\frac{dt}{\sqrt{1+t^3}}\qquad g'(x)=-\sin x\cdot(1+\sin(\cos x))^2$$

$$f'(x)=g'(x)\cdot\frac{1}{\sqrt{1+(g(x))^3}}$$

$$f'\left(\frac{\pi}{2}\right)=g'\left(\frac{\pi}{2}\right)=-\sin\frac{\pi}{2}=-1$$

72. $$x^2f(x)=\int_4^x(4t^2-2f'(t))\,dt$$

$$x^2f'(x)+2x\,f(x)=4x^2-2f'(x)$$

$$16f'(4)+8f(4)=64-2f'(4)$$

$$18f'(4)=64$$

$$9f'(4)=32$$

73. $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{2n}\frac{r^2}{n^3+r^3} = \lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{2n}\frac{\left(\frac{r}{n}\right)^2}{1+\left(\frac{r}{n}\right)^3} = \int_0^2\frac{x^2dx}{1+x^3} = \left|\frac{1}{3}\ln|1+x^3|\right|_0^2 = \frac{1}{3}\ln 9$

74. $\int_0^{2\pi}\cos^{-1}(\cos x)\,dx = 2\int_0^{\pi}\cos^{-1}(\cos x)\,dx = 2\cdot\left|\frac{x^2}{2}\right|_0^{\pi} = \pi^2$

75. $2f(x) = \int_0^x (x^2-2xt+t^2)\,g(t)\,dt$

$$2f(x) = x^2\int_0^x g(t)\,dt - 2x\int_0^x t\cdot g(t)\,dt + \int_0^x t^2 g(t)\,dt$$

$$2f'(x) = x^2\cdot g(x) + \int_0^x g(t)\,dt\cdot 2x - 2x(x\,g(x)) - \int_0^x t\cdot g(t)\,dt\cdot 2 + x^2 g(x)$$

$$2f'(x) = 2x\int_0^x g(t)\,dt - 2\int_0^x tg(t)\,dt$$

$$f''(x) = x\cdot g(x) + \int_0^x g(t)\,dt - xg(x)$$

$$f''(x) = \int_0^x g(t)\,dt$$

$$f'''(x) = g(x)$$

76. $I = \int_0^{\pi}\frac{x^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}dx = \lambda\int_0^{\pi/2}\sin^2 x\,dx$

$$I = \int_0^{\pi}\frac{(\pi-x)^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}dx \Rightarrow 2I = \int_0^{\pi}\pi\cos^4 x\sin^2 x\,dx$$

77. $\frac{1}{2}\cdot\left|\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right|_0^{\sqrt{3}} = \left|\tan^{-1}x\right|_0^{\sqrt{3}} = \frac{\pi}{3}-0 = \frac{\pi}{3}$

78. $\int_0^3\{x\}^{[x]}dx = \int_0^3(x-[x])^{[x]}dx = \int_0^1 1\cdot dx + \int_1^2(x-1)\,dx + \int_2^3(x-2)^2dx$

79. $I = \int_0^1\frac{\tan^{-1}x}{x}dx$

$x = \tan\theta$

$$I=\int_0^{\pi/4}\frac{\theta}{\tan\theta}\cdot\sec^2\theta\,d\theta=\int_0^{\pi/4}\frac{2\theta}{\sin 2\theta}\,d\theta=\frac{1}{2}\int_0^{\pi/2}\frac{t}{\sin t}\,dt$$

80. $$\int_0^{4/\pi}\underbrace{3x^2}_{\text{II}}\,\underbrace{\sin\frac{1}{x}}_{\text{I}}\,dx-\int_0^{4/\pi}x\cos\frac{1}{x}\,dx=\left|\sin\frac{1}{x}\cdot x^3\right|_0^{4/\pi}-\int_0^{4/\pi}\cos\frac{1}{x}\cdot\left(\frac{-1}{x^2}\right)\cdot x^3\,dx-\int_0^{4/\pi}x\cos\frac{1}{x}\,dx$$

$$\frac{64}{\pi^3}\cdot\frac{1}{\sqrt{2}}-\lim_{x\to0}\left(x^3\sin\frac{1}{x}\right)=\frac{32\sqrt{2}}{\pi^3}$$

81. $$\int_{-1}^{x}\left(8t^2+\frac{28t}{3}+4\right)dt=\frac{\frac{3x}{2}+1}{\log_{(x+1)}\sqrt{x+1}}$$

$$\left|\frac{8t^3}{3}+\frac{14t^2}{3}+4t\right|_{-1}^{x}=\frac{\frac{3x}{2}+1}{\frac{1}{2}}$$

$$\frac{8x^3}{3}+\frac{14x^2}{3}+4x-\left(\frac{-8}{3}+\frac{14}{3}-4\right)=3x+2$$

$$8x^3+14x^2+12x+8-14+12=9x+6$$

$$8x^3+14x^2+3x=0$$

$$x(8x^2+14x+3)=0$$

$$x(2x+3)(4x+1)=0$$

$$x=0,-\frac{3}{2},-\frac{1}{4}$$

But $x>-1\;\&\;x\neq0$

So, $x=-\frac{1}{4}$

85. $$f(x)=\int_0^4 e^{|x-t|}dt=\int_0^x e^{(x-t)}dt+\int_x^4 e^{(t-x)}dt=e^x+e^{4-x}-2\geq 2e^2-2$$

86. $$\frac{1}{4}\int_0^\infty\frac{4\cos^3 x}{x}dx=\frac{1}{4}\int_0^\infty\frac{\cos 3x+3\cos x}{x}dx=\frac{1}{4}\int_0^\infty\frac{\cos 3x}{x}dx+\frac{3}{4}\int_0^\infty\frac{\cos x}{x}dx$$

87. $$\int\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)dx=\int\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)dx=\int\cos x\,dx$$

88. $$I_{n+\frac{1}{2}} = \int_0^{\pi} \frac{\sin(2nx+x)}{\sin 2x}\,dx = \int_0^{\pi} \frac{\sin 2nx \cdot \cos x}{\sin 2x}\,dx + \int_0^{\pi} \frac{\cos 2nx \cdot \sin x}{\sin 2x}\,dx$$

$$= \frac{1}{2}\int_0^{\pi} \frac{\sin 2nx}{\sin x}\,dx + \frac{1}{2}\int_0^{\pi} \frac{\cos 2nx}{\cos x}\,dx$$

89. $f'(x) = 1 + \ln^2 x + 2\ln x = 0 \Rightarrow x = \frac{1}{e}$

$$f\left(\frac{1}{e}\right) = 1 + \frac{1}{e} + \int_1^{1/e} (\ln^2 t + 2\ln t)\,dt$$

Let $I = \int_1^{1/e} (\ln^2 t + 2\ln t)\,dt$

$$\ln t = x \Rightarrow t = e^x;\ dt = e^x dx = \int_0^{-1} (x^2 + 2x)\,e^x\,dx = [e^x \cdot x^2]_0^{-1} = \frac{1}{e}$$

90. $$f(x) = x^2 + \int_0^x e^{-t} f(x-t)\,dt$$

$$x^2 + \int_0^x e^{t-x} f(t)\,dt = x^2 + e^{-x}\int_0^x e^t f(t)\,dt$$

$$\Rightarrow \qquad f'(x) = 2x - e^{-x}\int_0^x e^t f(t)\,dt + f(x)$$

$$\Rightarrow \qquad f'(x) = 2x + x^2 \ \Rightarrow\ f(x) = \frac{x^3}{3} + x^2$$

$$\Rightarrow \qquad y = \frac{1}{4}(-2x^2 + 6x - 1)$$

91. $$I = \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+5^x}\,dx = \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+5^{-x}}\,dx$$

$$\Rightarrow\ 2I = \int_{-\pi/2}^{\pi/2} \cos^2 x\,dx$$

$$\Rightarrow\ I = \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}$$

92. $\int\left(\frac{x^2-x+1}{x^2+1}\right)e^{\cot^{-1}x}dx$

Let $\cot^{-1}x=1 \Rightarrow \frac{-1}{1+x^2}dx=dt$

$$\int e^t(\cot t-\operatorname{cosec}^2 t)\,dt=e^t\cdot\cot t+c$$

$$=x\cdot e^{\cot^{-1}x}+c$$

93. $\lim\limits_{x\to\infty}\frac{1}{n}\sum\limits_{r=1}^{n}\left(\frac{r\sqrt{n^2+r^2}}{n^2}\right)=\lim\limits_{x\to\infty}\frac{1}{n}\sum\limits_{r=1}^{n}\frac{r}{n}\sqrt{1+\left(\frac{r}{n}\right)^2}=\int\limits_0^1 x\sqrt{1+x^2}\,dx=\left[\frac{(1+x^2)^{3/2}}{3}\right]_0^1$

94. $\int\frac{(x^3-1)}{(x^4+1)(x+1)}dx=\int\frac{x^3}{x^4+1}dx-\int\frac{1}{x+1}dx=\frac{1}{4}\ln(1+x^4)-\ln(1+x)+c$

95. $\lim\limits_{x\to 0^+}\frac{(\cos^{-1}\cos x)(-\sin x)}{2-2\cos 2x}=\lim\limits_{x\to 0^+}\frac{-x\sin x}{4\sin^2 x}=-\frac{1}{4}$

96. $f(x)\begin{cases}\rightarrow 0 & x>\tan 1\\ \rightarrow \cos x & 0<x<\tan 1\\ \rightarrow \frac{\cos x}{2} & x=\tan 1\end{cases}$

$$\int\limits_0^{\infty}f(x)\,dx=\int\limits_0^{\tan 1}\cos x+\int\limits_{\tan 1}^{\infty}0\,dx=\sin(\tan 1)$$

97. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\frac{k}{n^2+n+2k}\right)=\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{k=1}^{n}\left(\frac{\frac{k}{n}}{1+\frac{1}{n}+\frac{2k}{n^2}}\right)=\int\limits_0^1 x\,dx$

98. $\lim\limits_{y\to 1^+}\frac{\int\limits_1^y|t-1|\,dt}{\tan(y-1)}\Rightarrow\lim\limits_{y\to 1^+}\frac{y-1}{\sec^2(y-1)}=0$ (Applying L' Hospital Rule)

99.

$$\int_0^1\frac{dx}{(1+x^2)^4}=\left[\frac{x}{2(4-1)(1+x^2)^{4-1}}\right]_0^1+\frac{5}{6}\int_0^1\frac{dx}{(1+x^2)^3}$$

$$=\left(\frac{1}{6(2)^3}-0\right)+\frac{5}{6}\left[\frac{x}{2(2)(1+x^2)^2}\right]_0^1+\frac{5}{6}\cdot\frac{3}{4}\int_0^1\frac{dx}{(1+x^2)^2}$$

$$= \frac{1}{48} + \left(\frac{5}{6}\right)\left[\frac{1}{16} - 0\right] + \frac{5}{8}\left[\frac{x}{2(1)(1+x^2)}\right]_0^1 + \frac{5}{8} \times \frac{1}{2}\int_0^1 \frac{dx}{1+x^2}$$

$$= \frac{1}{48} + \frac{5}{6 \times 16} + \frac{5}{8}\left(\frac{1}{4} - 0\right) + \frac{5}{16}[\tan^{-1} x]_0^1$$

$$= \frac{7}{6 \times 16} + \frac{5}{8 \times 4} + \frac{5}{16}\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{22}{6 \times 16} + \frac{5\pi}{64}$$

$$= \frac{11}{48} + \frac{5\pi}{64}$$

Alternate solution :

$$I = \int_0^1 \frac{dx}{(1+x^2)^4}$$

Put $x = \tan\theta$; therefore, $dx = \sec^2\theta\, d\theta$.

$$I = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(\sec\theta)^8}$$

That is,

$$I = \int_0^{\pi/4} (\cos\theta)^6\, d\theta$$

$$= \int_0^{\pi/4} \left(\frac{3\cos\theta + \cos 3\theta}{4}\right)^2 d\theta$$

$$= \frac{9}{16}\int_0^{\pi/4} \cos^2\theta\, d\theta + \frac{1}{16}\int_0^{\pi/4} (\cos 3\theta)^2\, d\theta + \frac{3}{8}\int_0^{\pi/4} \cos\theta \cos 3\theta\, d\theta$$

$$= \frac{9}{16}\int_0^{\pi/4} \frac{1+\cos 2\theta}{2}\, d\theta + \frac{1}{16}\int_0^{\pi/4} \frac{1+\cos 6\theta}{2}\, d\theta + \frac{3}{8}\int_0^{\pi/4} \frac{\cos 4\theta + \cos 2\theta}{2}\, d\theta$$

$$= \frac{9}{32}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} + \frac{1}{16 \times 2}\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi/4} + \frac{3}{8 \times 2}\left[\frac{\sin 4\theta}{4} + \frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= \left(\frac{9}{32}\right)\left[\frac{\pi}{4} + \frac{1}{2}\right] + \frac{1}{16 \times 2}\left[\frac{\pi}{4} - \frac{1}{6}\right] + \frac{3}{8 \times 2}\left[0 + \frac{1}{2} - 0\right]$$

$$= \frac{5}{64\pi} + \frac{11}{48}$$

100. We have,

$$I = \int_0^{\pi/4} (\sin x)^4\, dx \qquad \ldots(1)$$

We know that,

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Therefore,

$$\sin^4 x = (\sin x)^4 = \left(\frac{1-\cos 2x}{2}\right)^2$$

$$= \frac{1}{4}[1-2\cos 2x + (\cos 2x)^2]$$

$$= \frac{1}{4}\left(1-2\cos 2x + \frac{1+\cos 4x}{2}\right)$$

$$= \frac{1}{4}\left(\frac{3}{2}-2\cos 2x + \frac{\cos 4x}{2}\right)$$

Substituting this value of $\sin^4 x$ in Eq. (1), we get

$$I = \int_0^{\pi/4}\left(\frac{3}{8}-\frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right)dx$$

$$= \left[\frac{3}{8}x\right]_0^{\pi/4} - \frac{1}{4}[\sin 2x]_0^{\pi/4} + \frac{1}{32}[\sin 4x]_0^{\pi/4}$$

$$= \left(\frac{3}{8}\cdot\frac{\pi}{4}\right) - \frac{1}{4}(1-0) + \frac{1}{32}(0-0)$$

$$= \frac{3\pi}{32} - \frac{1}{4}$$

Alternate solution : We have,

$$I = \int_0^{\pi/4} (\sin x)^4\, dx$$

which can be written as

$$J = \int (\sin^2 x)(1-\cos^2 x)\, dx$$

$$= \int \sin^2 x\, dx - \frac{1}{4}\int 4\sin^2 x\cos^2 x\, dx$$

$$= \int \frac{1-\cos 2x}{2}\, dx - \frac{1}{4}\int (\sin 2x)^2\, dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x - \frac{1}{4}\int \frac{1-\cos 4x}{2}\, dx$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} - \frac{1}{8}x + \frac{1}{32}\sin 4x + c$$

$$= \frac{3}{8}x - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$$

Using the given limits, the above equation becomes

$$I=[J]_0^{\pi/4}=\left[\frac{3}{8}x\right]_0^{\pi/4}-\left[\frac{\sin 2x}{4}\right]_0^{\pi/4}+\left[\frac{\sin 4x}{32}\right]_0^{\pi/4}$$

$$=\frac{3\pi}{32}-\frac{1}{4}$$

101. $\int\frac{(\cos 9x+\cos 6x)\sin 5x}{\sin 10x-\sin 5x}\,dx=\int 2\cos\frac{5x}{2}\cos\frac{3x}{2}=\int(\cos 4x+\cos x)$

$$=\frac{\sin 4x}{4}+\sin x+C$$

$A=\frac{1}{4}, B=1$

102. $\int\frac{\frac{dx}{x^{2014}}}{1+\frac{1}{x^{2013}}}=\frac{1}{2013}\ln\left(\frac{x^{2013}}{1+x^{2013}}\right)+C$

103. $\frac{1}{2}\int_0^1 x\cdot(2x\cdot e^{-x^2})\,dx=\frac{1}{2}\left[\left(-xe^{-x^2}\right)+\int_0^1 e^{-x^2}dx\right]$

$$=\frac{1}{2}\left[-\frac{1}{e}+a\right]$$

104. $2\left[\int_0^1 f(x)\,dx+\int_1^2 f(x)\,dx+\int_2^3 f(x)\,dx\right]+\int_3^4 f(x)\,dx+\int_4^5 f(x)\,dx$

$$=2\left[\frac{0^2}{2}+\frac{1^2}{2}+\frac{2^2}{2}\right]+\frac{3^2}{2}+\frac{4^2}{2}=\frac{35}{2}$$

105. $\frac{1}{3}\int\frac{3x^2}{x^6(1+x^3)^2}dx$

Let $1+x^3=t\Rightarrow 3x^2dx=dt$

$$\Rightarrow\quad\frac{1}{3}\int\frac{dt}{t^2(t-1)^2}=\frac{1}{3}\int\left(\frac{2}{t}+\frac{1}{t^2}-\frac{2}{t-1}+\frac{1}{(t-1)^2}\right)dt$$

106. $\lim_{n\to\infty}\sum_{r=1}^{3n}\frac{1}{n\sqrt{1+\frac{r}{n}}}=\int_0^3\frac{1}{\sqrt{1+x}}dx=(2\sqrt{1+x})_0^3=2$

107. $\int_0^2 x\,f(x)\,dx=\left[\frac{x^2}{2}f(x)\right]_0^2-\int_0^2\frac{x^2}{2}f'(x)\,dx=0+\int_0^2\frac{x^2}{2\sqrt{1+x^3}}dx$

108. $\int_0^{\pi/3}(\ln(\cos x+\sqrt{3}\sin x)-\ln\cos x)\,dx$

$$=\int_0^{\pi/3}\left\{\ln\left(2\cos\left(x-\frac{\pi}{3}\right)\right)-\ln\cos x\right\}dx=\frac{\pi}{3}\ln 2$$

109. $\sum_{r=1}^{100}\int_0^1 f(r-1+x)\,dx=\int_0^1 f(x)\,dx+\int_0^1 f(x+1)\,dx+\int_0^1 f(x+2)\,dx+\ldots+\int_0^1 f(x+99)\,dx$

$$=\int_0^1 f(x)\,dx+\int_1^2 f(x)\,dx+\int_2^3 f(x)\,dx+\ldots+\int_{99}^{100} f(x)\,dx=\int_0^{100} f(x)\,dx=a$$

110. $\lim_{n\to\infty}\sum_{k=0}^{n} x^2\left(\frac{(2x)^k}{k!}\right)=x^2\cdot e^{2x}\Rightarrow\int_0^1 x^2e^{2x}\,dx=\frac{e^2-1}{4}$

111. $\int x^5\sqrt{1+x^3}\,dx$

Let $1+x^3=t^2$

$3x^2dx=2t\,dt$

$\frac{2}{3}\int t^2(t^2-1)\,dt=\frac{2}{3}\left(\frac{t^5}{5}-\frac{t^3}{3}\right)+c$

112. $f'(x)=\frac{\sin x}{x}$

$f'(x)>0\ \forall\ x\in(0,\pi)$

$f'(x)<0\ \forall\ x\in(\pi,2\pi)$

113. $\int\frac{x(x^2+1)+3(x^2+3)}{(x^2+1)(x^2+3)}dx$

$\int\left(\frac{x}{x^2+3}+\frac{3}{x^2+1}\right)dx$

$\frac{1}{2}\ln|x^2+3|+3\tan^{-1}x+c$

114. $\int\frac{\sqrt{\sec^5 x}}{\sqrt{\sin^3 x}}dx=\int\frac{\sec^4 x}{\sqrt{\tan^3 x}}dx$

Let $\tan x=t^2$

$\sec^2 x\,dx=2t\,dt$

$\int\frac{(1+t^4)\cdot 2t\cdot dt}{t^3}=2\int\left(\frac{1}{t^2}+t^2\right)dt$

115. Let $tx = y \Rightarrow x\,dt = dy$

$$\lim_{x\to 0}\frac{\int\limits_0^{x^2} e^{\sin y}\,dy}{x^2} = \lim_{x\to 0}\frac{e^{\sin x^2}\,2x}{2x} = 1$$

116. $$\int\limits_0^{\pi/2}\frac{\cos 2x}{x}\,dx = \left(\frac{\sin 2x}{2x}\right)_0^{\pi/2} + 2\int\limits_0^{\pi/2}\frac{\sin 2x}{(2x)^2}\,dx = -1 + \int\limits_0^{\pi}\frac{\sin\theta}{\theta^2}\,d\theta \qquad (\because \text{Let } 2x = \theta)$$

Exercise-2 : One or More than One Answer is/are Correct

1. $\displaystyle\int\frac{dx}{(1+\sqrt{x})^8}$ Let $x = t^2 \Rightarrow dx = 2t\,dt$

$$\int\frac{2t\,dt}{(1+t)^8} = 2\left[\int\frac{dt}{(t+1)^7} - \int\frac{dt}{(t+1)^8}\right] = 2\left[-\frac{1}{6(1+t)^6} + \frac{1}{7(1+t)^7}\right] + C$$

2. $$\int\limits_{-\alpha}^{\alpha}\left[e^x + \cos x\ln\left(x+\sqrt{1+x^2}\right)\right]dx = 2\int\limits_0^{\alpha} e^x\,dx = 2(e^{\alpha}-1) \Rightarrow e^{\alpha} > \frac{7}{4}$$

3. $\displaystyle I = \int\sqrt{\frac{x}{a^3-x^3}}\,dx$ Let $x^{3/2} = a^{3/2}\cos\theta$

$$= \int\frac{\sqrt{x}}{\sqrt{(a^{3/2})^2-(x^{3/2})^2}}\,dx = \frac{2}{3}\int\frac{a^{3/2}\cos\theta}{\sqrt{a^3-a^3\sin^2\theta}}\,d\theta$$

$$= \frac{2}{3}\int d\theta = \frac{2}{3}\theta + C = \frac{2}{3}\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C$$

4. $$\int x\sin x\sec^3 x\,dx = \int\underset{\text{I}}{x}\left(\underbrace{\tan x\sec^2 x}_{\text{II}}\right)dx = x\frac{\tan^2 x}{2} - \int\frac{\tan^2 x}{2}\,dx$$

$$= x\frac{\tan^2 x}{2} - \int\frac{(\sec^2 x - 1)}{2}\,dx = x\frac{\tan^2 x}{2} - \frac{1}{2}(\tan x - x) + C$$

$$= \frac{1}{2}(x\sec^2 x - \tan x) + C$$

$\therefore\quad f(x) = \sec^2 x,\quad g(x) = \tan x$

(a) Clear $f(x) \notin (-1, 1)$

(b) $\tan x = \sin x$

$\Rightarrow \quad \cos x = 1 \;\Rightarrow\; \tan x$ is not defined.

no solution

(c) $g'(x) = f(x) \;\forall\; x \in R$ except $(2n-1)\dfrac{\pi}{2}$

(d) $\sec^2 x = \tan x$

$1 + \tan^2 x - \tan x = 0$ has no solution.

5. $\displaystyle\int (\sin 3\theta + \sin\theta)\cos\theta\, e^{\sin\theta} d\theta = \int (4\sin\theta - 4\sin^3\theta)\, e^{\sin\theta} \cos\theta\, d\theta$

Let $t = \sin\theta$

$dt = \cos\theta\, d\theta = 4[-t^3 + 3t^2 - 5t + 5]e^t + C$

Compare it

$$A = -4, B = -12, C = -20$$

7. $\displaystyle I = \int_0^{\theta} \frac{2x\,dx}{\sqrt{(3\theta - 2x)(\theta + 2x)}} = \int_0^{\theta} \frac{2(\theta - x)\,dx}{\sqrt{(3\theta - 2x)(\theta + 2x)}}$

$\displaystyle\Rightarrow \quad I = \frac{\theta}{2}\int_0^{\theta} \frac{dx}{\sqrt{\theta^2 - (x - \theta/2)^2}}$

8. Let $f(x) = a^x, F(x) = F(-x)$

9. $\displaystyle J = \int_{-1}^{0} \left[\cot^{-1}\left(\frac{1}{x}\right) + \cot^{-1}(x)\right] dx + \int_0^2 \left(\cot^{-1}\left(\frac{1}{x}\right) + \cot^{-1} x\right) dx$

$\displaystyle = \int_{-1}^{0} \left(\pi + \frac{\pi}{2}\right) dx + \int_0^2 \frac{\pi}{2}\, dx$

$\displaystyle K = \int_0^{\pi} dx = \pi \qquad$ (As 2π is period)

11. $\displaystyle l_1 = \lim_{x\to\infty} \sqrt{\frac{x - \cos^2 x}{x + \sin x}} = 1$

$\displaystyle l_2 = \lim_{h\to 0^+} \int_{-1}^{1} \frac{h\,dx}{h^2 + x^2} = \lim_{h\to 0} 2\tan^{-1}\frac{1}{h} = \pi$

13. $\displaystyle\int \frac{dx}{(1 + \sin^2 x)\cos^2 x} = \int \frac{\sec^4 x}{1 + 2\tan^2 x}\, dx$

$$= \int \frac{(1+\tan^2 x)\sec^2 x}{(1+2\tan^2 x)} dx = \frac{1}{2}\int \sec^2 x \, dx + \frac{1}{2}\int \frac{\sec^2 x \, dx}{1+2\tan^2 x}$$

$$= \frac{1}{2}\tan x + \frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + c$$

14. $\int \frac{(1+\sin^{2015} x) - \sqrt{1+\sin^{4030} x}}{2\sin^{2015} x}$ (Rationalise)

$\int_{-2014}^{2014} \frac{1}{2} dx$ $\int \text{odd} = 0$

15. $\tan^{-1}(nx)\big|_a^\infty = \frac{\pi}{2} - \tan^{-1}(na)$

$a > 0, a = 0, a < 0$

16. Let $\sqrt{x} = \cos 2\theta$

$dx = -\sin 4\theta \, d\theta$

$I = \int_0^{\pi/4} \cot\theta \sin 4\theta \, d\theta$ and $J = \int_0^{\pi/4} \tan\theta \sin 4\theta \, d\theta$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

2. $f(x) = \int (2x^3\cos^2 x + 6x^2 \sin x \cos x - 2x^3 \sin^2 x)\, dx$

$$= \int \left(\underbrace{2x^3}_{\text{I}} \underbrace{\cos 2x}_{\text{II}} + 3x^2 \sin 2x \right) dx$$

$f(x) = x^3 \sin 2x + c$

$f(\pi) = 0 + c = 0 \Rightarrow c = 0$

$f(x) = x^3 \sin 2x$

Paragraph for Question Nos. 6 to 8

6. $g(x) = x - A$

$A = \int_0^1 f(t)\, dt$

$$f(x)=\frac{x^3}{2}+1-x\int_0^x (x-A)\,dx=\frac{x^3}{2}+1-\frac{x^3}{2}+Ax^2$$

$$f(x)=Ax^2+1$$

$$A=\int_0^1 Ax^2+1 \;\Rightarrow\; A=\frac{3}{2}$$

$$f(x)=\frac{3x^2}{2}+1;\;\; \text{min. } f(x)=1$$

7. $\frac{3}{2}x^2+1=x-\frac{3}{2}$

$3x^2-2x+5=0$

$\Delta<0$, no solution

8. $g(x)=x-\frac{3}{2}$

$$A=\frac{1}{2}\times\frac{3}{2}\times\frac{3}{2}=\frac{9}{8}$$

Paragraph for Question Nos. 9 to 11

9.
$$\int_0^a f(x)\,dx-\int_a^1 f(x)\,dx=2f(a)+3a+b \qquad \text{...(1)}$$

Diff. w.r.t. 'a' on both sides,

$$(f(a)-0)-(0-f(a))=2f'(a)+3$$

$$2f(a)=2f'(a)+3$$

$$(2f(a)-3)=2f'(a)$$

$$\frac{2f'(a)}{2f(a)-3}=1$$

$$\int\frac{2f'(a)}{2f(a)-3}\,da=\int da$$

$$\ln|2f(a)-3|=a+c$$

$$2f(a)-3=e^{a+c}$$

$$2f(a)-3=ke^{a}$$

$$2f(a)=ke^{a}+3$$

Put $a = 1 \Rightarrow 0 = ke + 3 \Rightarrow k = -\frac{3}{e}$

$$\therefore \quad 2f(a) = -\frac{3}{e}e^{a} + 3$$

$$f(a) = \frac{3}{2} - \frac{3}{2e}e^{a}$$

$$f(x) = \frac{3}{2} - \frac{3}{2e}e^{x}$$

Put $f(x)$ in (1) (By taking limiting case)

$$\int_{0}^{a}\left(\frac{3}{2} - \frac{3}{2e}e^{x}\right)dx - \int_{a}^{1}\left(\frac{3}{2} - \frac{3}{2e}e^{x}\right)dx = 3 - \frac{3}{e}e^{a} + 3a + b$$

$$\left[\left(\frac{3a}{2} - \frac{3}{2e}e^{a}\right) - \left(0 - \frac{3}{2e}\right)\right] - \left[\left(\frac{3}{2} - \frac{3}{2}\right) - \left(\frac{3a}{2} - \frac{3e^{a}}{2e}\right)\right] = 3 - \frac{3}{e}e^{a} + 3a + b$$

$$\frac{3}{2e} - 3 = b$$

10. Length of subtangent $= \left|\frac{y}{(dy/dx)}\right|$

$$y = f(x) = \frac{3}{2} - \frac{3}{2e}e^{x}$$

$$\frac{dy}{dx} = f'(x) = 0 - \frac{3}{2e}e^{x}$$

$$\left.\frac{dy}{dx}\right|_{x=1/2} = -\frac{3}{2\sqrt{e}}$$

when $x = \frac{1}{2}$, $y = f\left(\frac{1}{2}\right) = \frac{3}{2} - \frac{3}{2e}e^{1/2} = \frac{3}{2}\left(1 - \frac{1}{\sqrt{e}}\right)$

$$\text{Length of subtangent} = \left|\frac{\frac{3}{2}\left(1 - \frac{1}{\sqrt{e}}\right)}{-\frac{3}{2\sqrt{e}}}\right| = \left|\sqrt{e} - 1\right| = \sqrt{e} - 1$$

11. $\int_{0}^{1} f(x)\,dx = \int_{0}^{1}\left(\frac{3}{2} - \frac{3}{2e}e^{x}\right)dx = \left.\frac{3x}{2} - \frac{3}{2e}e^{x}\right|_{0}^{1} = \left(\frac{3}{2} - \frac{3e}{2e}\right) - \left(0 - \frac{3}{2e}\right)$

$$= \left(\frac{3}{2} - \frac{3}{2}\right) + \frac{3}{2e} = \frac{3}{2e}$$

Paragraph for Question Nos. 12 to 13

12. $f_3'''(x) = f_0(x)$ see options or 3 times by parts.

13. $f_n(x) = \dfrac{x^n}{\underline{|n}}\left(\ln x - 1 - \dfrac{1}{2} - \dfrac{1}{3} \dots - \dfrac{1}{n}\right)$

Paragraph for Question Nos. 14 to 15

$$f(x) = a\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$f(1) = 1 \qquad a = 1$

$f(x) = x^2 - x + 1$

14. $g(x) = 1 + x^2$ now integrate

15. $\displaystyle\int \frac{e^x}{e^{2x} - e^x + 1} \qquad e^x = t$

Paragraph for Question Nos. 16 to 17

17. $$L = \lim_{x\to\infty} \frac{xe^{2x}(1+3x^2)^{1/2}}{C\cdot(xe^x)^{C-1}\cdot(e^x + xe^x)} = \lim_{x\to\infty} \frac{(xe^x)^2\left(\dfrac{1}{x^2}+3\right)^{1/2}}{C\cdot(xe^x)^C\cdot\left(\dfrac{1}{x}+1\right)}$$

Exercise-4 : Matching Type Problems

2. (A) $$\int \frac{dx}{(x^2+1)\sqrt{x^2+2}} = \int \frac{\dfrac{1}{x^3}\,dx}{\left(1+\dfrac{1}{x^2}\right)\sqrt{1+\dfrac{2}{x^2}}}$$

Let $1 + \dfrac{2}{x^2} = t^2$

(C) $$\int \frac{x^4 + x^8}{(1-x^4)^{7/2}} = \int \frac{\left(x + \dfrac{1}{x^3}\right)dx}{\left(\dfrac{1}{x^2} - x^2\right)^{7/2}}$$

Let $\dfrac{1}{x^2} - x^2 = t^2$

(D) Let $\sqrt{x} = \cos 2\theta \Rightarrow dx = -2\sin 4\theta \, d\theta$

$$\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx = -2\int \tan\theta \sin 4\theta \, d\theta$$

3. (A) Let $\sin x = t \quad \Rightarrow \cos x\, dx = dt$

$$\int_0^1 \frac{dt}{(1+t)(2+t)} = \int_0^1 \left(\frac{1}{t+1} - \frac{1}{t+2}\right) dt = [\ln(1+t) - \ln(t+2)]_0^1$$

(B) $$\int_0^{41\pi/4} |\cos x|\, dx = 10\int_0^{\pi} |\cos x|\, dx + \int_0^{\pi/4} \cos x\, dx$$

$$= 10\left[\int_0^{\pi/2} \cos x\, dx - \int_{\pi/2}^{\pi} \cos x\, dx\right] + \int_0^{\pi/4} \cos x\, dx$$

(C) $$\int_{-1/2}^{0} [x]\, dx + \int_0^{1/2} [x]\, dx + \int_{-1/2}^{1/2} \ln\left(\frac{1+x}{1-x}\right) dx$$

$$= \int_{-1/2}^{0} -1\, dx + \int_0^{1/2} 0\, dx = -\frac{1}{2}$$

(D) $$I = \int_0^{\pi/2} \frac{2\sqrt{\cos\theta}}{3(\sqrt{\cos\theta} + \sqrt{\sin\theta})}\, d\theta = \int_0^{\pi/2} \frac{2\sqrt{\sin\theta}}{3(\sqrt{\sin\theta} + \sqrt{\cos\theta})}\, d\theta$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{2}{3}\, d\theta = \frac{\pi}{3}$$

$$\Rightarrow \quad I = \frac{\pi}{6}$$

4. (A) Common root $\alpha = b - a \Rightarrow 3(b-a)^2 + a(b-a) + 1 = 0 \Rightarrow 2a^2 + 3b^2 - 5ab + 1 = 0$

(B) $$\frac{x^4+1}{2x^2} = \sin^2\frac{\pi x}{2} \Rightarrow \frac{x^2 + \frac{1}{x^2}}{2} = \sin^2\frac{\pi x}{2} \Rightarrow x = \pm 1$$

(C) $$y = \frac{1}{\frac{1}{(x-1)^2} + \frac{1}{x-1} - 2} \qquad x \neq 1 \;;\; \frac{1}{x-1} \neq -2, 1$$

(D) $$\int \left(\frac{x}{1+x}\right)^{7/6} \frac{dx}{x^2}.$$

Let $\frac{x+1}{x}=t^6 \Rightarrow -\frac{1}{x^2}dx=6t^5dt$

$$\therefore \quad I=6\int t^{-7}(-t^5)\,dt=\frac{6}{t}+C=6\left(\frac{x}{x+1}\right)^{1/6}+C$$

5. (A) We have,

$$\int_0^{1.5}[x^2]\,dx=\int_0^1[x^2]\,dx+\int_1^{\sqrt{2}}[x^2]\,dx+\int_{\sqrt{2}}^{1.5}[x^2]\,dx$$

$$=\int_0^1 0\,dx+\int_1^{\sqrt{2}}1\cdot dx+\int_{\sqrt{2}}^{1.5}2\cdot dx=0+(\sqrt{2}-1)+2(1.5-\sqrt{2})=2-\sqrt{2}$$

(B) We have,

$$\int_0^4\{\sqrt{x}\}\,dx=\int_0^1\sqrt{x}\,dx+\int_1^4(\sqrt{x}-1)\,dx=\frac{2}{3}+\frac{2}{3}(8-1)-3=\frac{7}{3}$$

Aliter : $\int_0^4\{\sqrt{x}\}\,dx=\int_0^4\sqrt{x}\,dx-\int_0^4[\sqrt{x}]\,dx$

(C) We have,

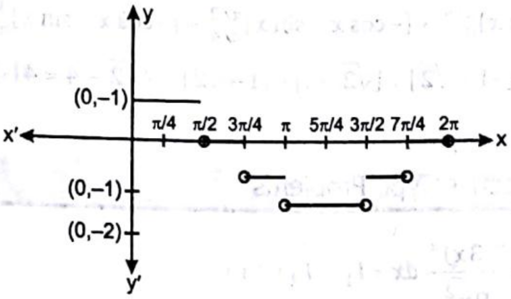
$$\sin x+\cos x=\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)$$

$$\therefore \quad [\sin x+\cos x]=\left[\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)\right]$$

The graph of $y=\left[\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)\right]$ is obtained from the graph of $y=[\sqrt{2}\sin x]$ by translating it by $\frac{\pi}{4}$ units in the direction of OX'. The graph so obtained is shown in figure.

It is evident from the graph of $y=[\sqrt{2}\sin(x+\pi/4)]$ that

$$f(x)=[\sin x+\cos x]=\begin{cases}1, & 0\le x\le \pi/2\\ 0, & \pi/2<x\le 3\pi/4\\ -1, & 3\pi/4<x\le \pi\\ -2, & \pi<x<3\pi/2\\ -1, & 3\pi/2\le x<7\pi/4\\ 0, & 7\pi/4\le x<2\pi\end{cases}$$



$$\therefore \quad \int_0^{2\pi} [\sin x + \cos x]\, dx$$

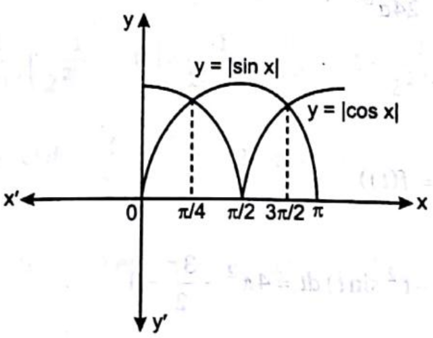
$$= \int_0^{\pi/2} 1 \cdot dx + \int_{\pi/2}^{3\pi/4} 0\, dx + \int_{3\pi/4}^{\pi} (-1)\, dx + \int_{\pi}^{3\pi/2} (-2)\, dx + \int_{3\pi/2}^{7\pi/4} (-1)\, dx + \int_{7\pi/4}^{2\pi} 0\, dx$$

$$= \frac{\pi}{2} + 0 - \frac{\pi}{4} - 2 \times \frac{\pi}{2} + (-1)\frac{\pi}{4} + 0 = -\pi$$

(D) We have,

$$\int_0^{\pi} \big|\,|\sin x| - |\cos x|\,\big|\, dx$$

$$= \int_0^{\pi/4} (|\sin x| - |\cos x|)\, dx + \int_{\pi/4}^{3\pi/4} (|\sin x| - |\cos x|)\, dx + \int_{3\pi/4}^{\pi} -(|\sin x| - |\cos x|)\, dx$$



$$= -\int_0^{\pi/4} (\sin x - \cos x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx + \int_{\pi/2}^{3\pi/4} (\sin x + \cos x)\, dx$$

$$+ \int_{3\pi/4}^{\pi} -(\sin x + \cos x)\, dx$$

$$=-[-\cos x-\sin x]_0^{\pi/4}+[-\cos x-\sin x]_{\pi/4}^{\pi/2}+[-\cos x-\sin x]_{\pi/2}^{3\pi/4}-[-\cos x+\sin x]_{3\pi/4}^{\pi}$$

$$=-[-\sqrt{2}+1]+[-1+\sqrt{2}]+[\sqrt{2}-1]-[1-\sqrt{2}]=4\sqrt{2}-4=4(\sqrt{2}-1)$$

Exercise-5 : Subjective Type Problems

1. $\displaystyle\int\frac{x\,dx}{\sqrt{1-9x^2}}+\int\frac{(\cos^{-1}3x)^2}{\sqrt{1-9x^2}}dx=I_1+I_2$

$$I_1=\int\frac{x\,dx}{\sqrt{1-9x^2}} \qquad I_2=\int\frac{(\cos^{-1}3x)^2}{\sqrt{1-9x^2}}dx$$

Let $1-9x^2=t^2$ Let $\cos^{-1}3x=k$

2. $\displaystyle I=\int_0^{\infty}\frac{x^3\,dx}{(a^2+x^2)^5}$

$$I=\frac{1}{a^6}\int_0^{\pi/2}\sin^3\theta\cos^5\theta\,d\theta=\frac{1}{a^6}\int_0^{\pi/2}\cos^3\theta\sin^5\theta\,d\theta \qquad (\text{Let } x=a\tan\theta)$$

$$2I=\frac{1}{8a^6}\int_0^{\pi/2}\sin^3 2\theta\,d\theta=\frac{1}{32a^6}\int_0^{\pi/2}(3\sin 2\theta-\sin 6\theta)\,d\theta$$

$$\Rightarrow \qquad I=\frac{1}{24a^6}$$

3. $\displaystyle\int_0^{2\pi}g(x)\,dx$

$$\int_{3\pi/2}^{2\pi}t\,f'(t)\,dt \qquad (\because x=f(t))$$

$$=\int_{3\pi/2}^{2\pi}(t\cos t-t^2\sin t)\,dt=4\pi^2-\frac{3\pi}{2}-1$$

4. $\displaystyle\int(x^5+x^3+x)\sqrt{2x^6+3x^4+6x^2}\,dx$

Let $2x^6+3x^4+6x^2=t^2 \Rightarrow 12(x^5+x^3+x)\,dx=2t\,dt$

$$=\frac{1}{12}\int 2t^2dt=\frac{1}{18}(2x^6+3x^4+6x^2)^{3/2}+C$$

5. Put $x = \sin\theta$

6. $\displaystyle\int \frac{\tan x}{\tan^2 x + \tan x + 1}\, dx$

Let $\tan x = t, \sec^2 x\, dx = dt,\ dx = \dfrac{dt}{1+t^2}$

$$\int \frac{t}{(1+t+t^2)} \cdot \frac{dt}{1+t^2} = \int \left(\frac{1}{1+t^2} - \frac{1}{1+t+t^2} \right) dt = \int \frac{dt}{1+t^2} - \int \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \tan^{-1}(t) - \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C$$

$$= \tan^{-1}(\tan x) - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2\tan x + 1}{\sqrt{3}}\right) + C$$

7. Let $x^4 = t$

$4x^3 dx = dt$

$$\int_0^1 \frac{1+t^{2010}}{(1+t)^{2012}}\, dt = \int_0^1 \frac{1}{(1+t)^{2012}}\, dt + \int_0^1 \frac{1}{t^2\left(1+\frac{1}{t}\right)^{2012}}\, dt = \left.\frac{(1+t)^{-2011}}{-2011}\right|_0^1 + \left.\frac{\left(1+\frac{1}{t}\right)^{-2011}}{2011}\right|_0^1$$

$$\frac{-1}{2011}\left(\frac{1}{2^{2011}} - 1\right) + \frac{1}{2011}\left(\frac{1}{2^{2011}} - 0\right) = \frac{-1}{2011}\left(\frac{1}{2^{2011}} - 1 - \frac{1}{2^{2011}}\right) = \frac{1}{2011} = \frac{\lambda}{\mu}$$

8. $\displaystyle\int_1^{\sqrt{3}} (x^{2x^2} x + 2x^{2x^2} \cdot x \ln x)\, dx = \int_1^{\sqrt{3}} x^{2x^2} (x + 2x \ln x)\, dx$

$\displaystyle\int_1^{\sqrt{3}} \left(x^{x^2}\right)^2 (x + 2x\ln x)\, dx$ Let $x^{x^2} = t \Rightarrow x^2 \ln x = \ln t\ ;\ (2x \ln x + x)\, dx = \dfrac{dt}{t}$

$$\int_1^{(\sqrt{3})^3} t^2 \cdot \frac{dt}{t} = \int_1^{(\sqrt{3})^3} t\, dt = \left.\frac{t^2}{2}\right|_1^{3^{3/2}} = \frac{3^3 - 1}{2} = 13$$

9. $\displaystyle\int \frac{dx}{(\cos x - \sin x)(1 + \sin x \cos x)} = 2\int \frac{(\cos x - \sin x)\, dx}{(\cos x - \sin x)^2 (2 + (\sin x + \cos x)^2 - 1)}$

$$= 2\int \frac{(\cos x - \sin x)\, dx}{((\sin^2 x + \cos^2 x) - 2\sin x \cos x)(1 + (\sin x + \cos x)^2)}$$

$$=2\int\frac{(\cos x-\sin x)\,dx}{(2-(\sin x+\cos x)^2)(1+(\sin x+\cos x)^2)}$$

$$=2\int\frac{dt}{(2-t^2)(1+t^2)}\text{ where } t=\sin x+\cos x$$

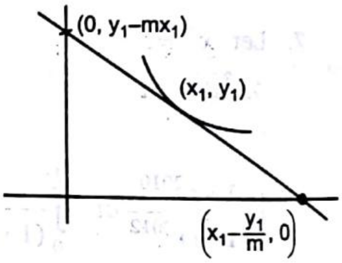
$$=2\int\frac{dt}{(2-t^2)(1+t^2)}=\frac{2}{3}\left[\int\frac{1}{1+t^2}dt+\int\frac{dt}{2-t^2}\right]$$

$$=\frac{2}{3}\left[\tan^{-1}(t)+\frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+t}{\sqrt{2}-t}\right)\right]+C$$

$$\therefore\quad A=\frac{2}{3},\ B=\frac{1}{3\sqrt{2}}$$

$$\therefore\quad 12A+9\sqrt{2}B-3=12\cdot\frac{2}{3}+9\sqrt{2}\,\frac{1}{3\sqrt{2}}-3=8$$

10. $x^a\cdot y=\lambda^a;\quad y=\dfrac{\lambda^a}{x^a}$



$$\frac{dy}{dx}=-a\lambda^a x^{-a-1}=-a\frac{x^a\cdot y}{x^{a+1}}$$

$$\Rightarrow\quad m=\frac{-ay_1}{x_1}$$

$$A=\frac{1}{2}|y_1-mx_1|\left|x_1-\frac{y_1}{m}\right|=\frac{1}{2}y_1x_1(1+a)^2$$

$$=\frac{1}{2}\lambda^a\cdot x_1^{1-a}(1+a)^2$$

For A to be constant $1-a=0$

11. $I_{(6,8)}=\displaystyle\int_0^{\pi}x^6(\pi-x)^8dx=\left(-\frac{x^6(\pi-x)^9}{9}\right)_0^{\pi}+\int_0^{\pi}6x^5\frac{(\pi-x)^9}{9}dx$

$$I_{(6,8)}=\frac{6}{9}I_{(5,9)}=\frac{6}{9}\times\frac{5}{10}\times\frac{4}{11}\times\frac{3}{12}\times\frac{2}{13}\times\frac{1}{14}\int_0^{\pi}(\pi-x)^{14}dx=\frac{6!\times8!}{15!}\pi^{15}$$

14. $I=\displaystyle\int_0^{100}\sqrt{x}\,dx-\left[\int_0^{1^2}0\cdot dx+\int_{1^2}^{2^2}dx+\ldots\ldots+\int_{2^2}^{3^2}2dx+\ldots\ldots+\int_{9^2}^{10^2}9\,dx\right]$

$$I=\frac{155}{3}$$

17. $f(\theta)=\int_{-1}^{1}\frac{\sin\theta\,dx}{(x-\cos\theta)^2+\sin^2\theta}=\tan^{-1}\left(\frac{x-\cos\theta}{\sin\theta}\right)\Bigg|_{-1}^{1}$

Clearly, $f(\theta)$ is not defined when $\sin\theta=0$

$\theta=0,\pi,2\pi$

20. $f(x)=\frac{1}{2}\int_0^x (x-t)^2 g(t)\,dt=\frac{1}{2}\left[x^2\int_0^x g(t)\,dt-2x\int_0^x tg(t)\,dt+\int_0^x t^2 g(t)\,dt\right]$

$$f'(x)=\left[x\int_0^x g(t)\,dt-\int_0^x t\,g(t)\,dt\right]$$

$$f''(x)=\int_0^x g(t)\,dt$$

$$f'''(x)=g(x)$$

22. $f(2-x)=f(2+x)$, it means it symmetric about $x=2 \Rightarrow \int_0^2 f(x)\,dx=\int_2^4 f(x)\,dx=5$

Let $2-x=t$; $f(t)=f(4-t)$ *i.e.*, $f(x)=f(4-x)=f(4+x)$

$$\int_0^{50} f(x)\,dx=25\left(\int_0^2 f(x)\,dx\right)=25\times 5=125$$

23. $I_n=\int_{-1}^{1}|x|\left(1+x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots\ldots+\frac{x^{2n}}{2n}\right)dx=2\left[\int_0^1\left(x+\frac{x^3}{2}+\frac{x^5}{4}+\ldots\ldots+\frac{x^{2n+1}}{2n}\right)dx\right]$

$$=2\left[\frac{x^2}{1\cdot 2}+\frac{x^4}{2\cdot 4}+\frac{x^6}{4\cdot 6}+\ldots\ldots+\frac{x^{2n+1}}{2n\cdot(2n+2)}\right]_0^1$$

$$=2\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 4}+\frac{1}{4\cdot 6}+\ldots\ldots+\frac{1}{2n\cdot(2n+2)}\right]$$

$$I_n=1+\frac{1}{2}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots\ldots+\frac{1}{n(n+1)}\right)=1+\frac{1}{2}\left(\frac{1}{1}-\frac{1}{n+1}\right)$$

$$\lim_{n\to\infty} I_n=1+\frac{1}{2}=\frac{3}{2}=\frac{p}{q}$$

$$pq(p+q)=3\times 2(5)=30$$

25. $\int_a^b|\sin x|\,dx=8 \quad \Rightarrow b-a=4\pi$

$$\int_0^{a+b} |\cos x|\,dx = 9 \quad \Rightarrow a+b=\frac{9\pi}{2} \Rightarrow a=\frac{\pi}{4};\, b=\frac{17\pi}{4}$$

$$\frac{1}{\sqrt{2}\pi}\left|\int_a^b x\sin x\cdot dx\right| = \frac{1}{\sqrt{2}\pi}\left|\int_{\pi/4}^{17\pi/4} x\sin x\,dx\right| = 2$$

28. $f(x)=0 \;\Rightarrow \displaystyle\int_0^x e^{-y} f'(y)\,dy = x^2 - x + 1$

$$\Rightarrow e^{-x} f'(x) = 2x-1 \;\Rightarrow f(x) = (2x-3)e^x$$

29. $I_n = 2\displaystyle\int_0^{\pi}\left(\frac{\pi}{2}-|x|\right)\cos nx\,dx = 2\left(\frac{1-\cos n\pi}{n^2}\right)$

$$I_1 + I_2 + I_3 + I_4 = 4\left(1+\frac{1}{9}\right) = \frac{40}{9}$$

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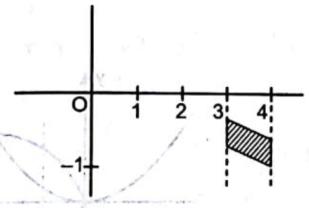
Chapter 6 – Area Under Curve

6

AREA UNDER CURVES

Exercise-1 : Single Choice Problems

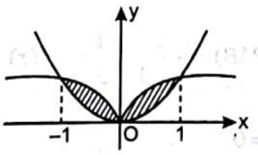
1.



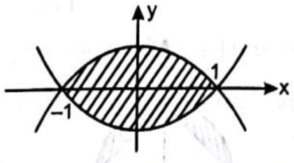
$1 \le x + 3y < 2$

Area of shaded region $= \dfrac{1}{3}$

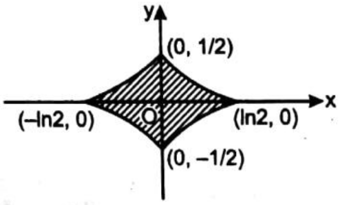
2. Area $= 2\displaystyle\int_0^1 (\sqrt{x} - x^2)\,dx$

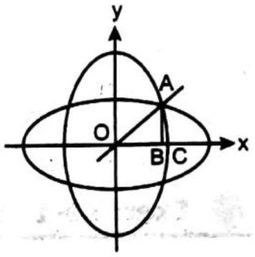


3. $y^2 = (x^2 - 1)^2 = 4\displaystyle\int_0^1 (1 - x^2)\,dx = \dfrac{8}{3}$



4. Area $= 4\displaystyle\int_0^{\ln 2} \left(e^{-x} - \dfrac{1}{2}\right) dx = 2 - 2\ln 2$



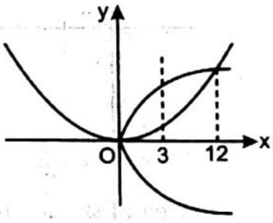


5. As given relations are inverse of each other so A lies on $y = x$

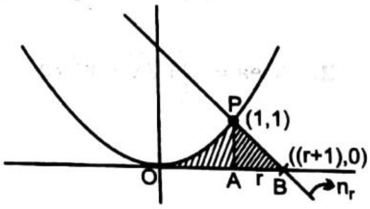
i.e., $\left(\dfrac{n}{\sqrt{n^2+1}}, \dfrac{n}{\sqrt{n^2+1}}\right)$

So, required area = 8 area ($OACBO$) = 8 (ΔOAB + area $BACB$)

$$= 8\left(\frac{1}{2}\left(\frac{n}{\sqrt{n^2+1}}\right)^2 + \int_{n/\sqrt{n^2+1}}^{1} n\cdot\sqrt{1-x^2}\,dx\right)$$



6. $$\frac{\displaystyle\int_0^3\left(12x-\frac{x^2}{12}\right)dx}{\displaystyle\int_3^{12}\left(12x-\frac{x^2}{12}\right)dx} = \frac{15}{49}$$



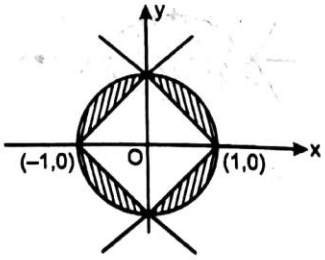
8. $$\left.\frac{dy}{dx}\right|_{(1,1)} = r$$

Normal at (1, 1) is $\Rightarrow y = -\dfrac{1}{r}(x-1)+1$

Required area $= \displaystyle\int_0^1 x^r\,dx + \text{ar}(\Delta PAB) = \frac{1}{r+1} + \frac{1}{2}r = f(r)$

$$f'(r) = -\frac{1}{(r+1)^2} + \frac{1}{2} = 0$$

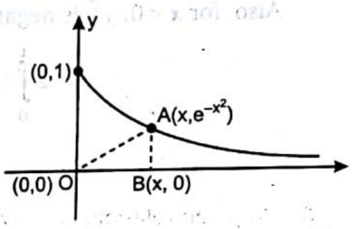
9. Area $= 4\left[\displaystyle\int_0^1 \sqrt{1-x}\,dx - \frac{1}{2}\right] = \frac{2}{3}$



10. Area of $\Delta AOB = \frac{1}{2} x e^{-x^2}$

$$\frac{dA}{dx} = (1-2x^2)\, e^{-x^2}$$

Area is maximum at $x = \frac{1}{\sqrt{2}}$



12. $\int_1^2 g(x)\, dx$

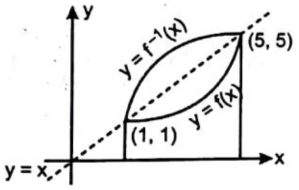
Let $x = f(t) \Rightarrow dx = f'(t)\, dt$

$$\int_0^1 t(3t^2 - 6t + 3)\, dt = \frac{1}{4}$$

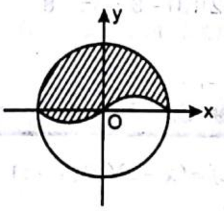
13. $x^2y^2 + y^4 - x^2 - 5y^2 + 4 = 0$

$(x^2 + y^2 - 4)(y^2 - 1) = 0$

14.



15. Ar. of shaded region $= \frac{1}{2}$ Ar. of circle $= \frac{\pi^3}{2}$.



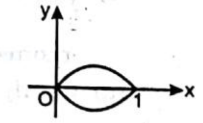
16. We have,

$$y^2 = x(1-x^3) \qquad \text{...(1)}$$

For $x > 1$, y^2 is negative. Since the square of a real number cannot be negative, y does not exist at $x = 0$ or at $x = 1$; $y = 0$. Let $x = \frac{1}{2}$. Therefore, from Eq. (1), we get

$$y^2 = \frac{1}{2}\left(1 - \frac{1}{8}\right) = \frac{7}{16}$$

$$y = \pm\frac{\sqrt{7}}{4}$$

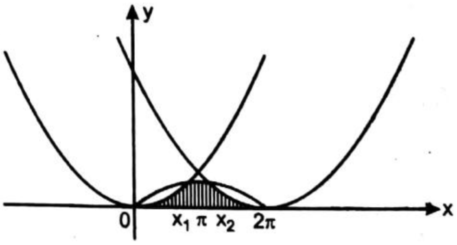


Also, for $x < 0$, y^2 is negative. Therefore, the required area is

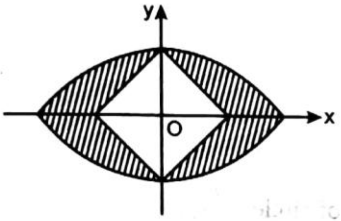
$$2\int_0^1 y\,dx = 2\int_0^1 (+)\sqrt{x}\sqrt{1-x^3}\,dx$$

$$= 2\int_0^1 \sqrt{x-x^4}\,dx$$

17.



18. $|x| + |y| \geq 2$ and $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$



Ar. of ellipse – Ar. of square $= \pi(2)(3) - 8 = 6\pi - 8$

Exercise-2 : One or More than One Answer is/are Correct

1. (a) $f(x) = -(x-a)(x-b)(x+c) = (x-a)(x)(x+c)$ $[\because b = 0]$

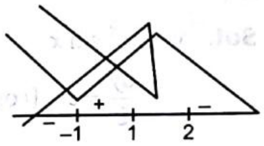
Clearly option (a) is correct.

(b) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx > 0$ (from graph)

which incorrect

(c) $\int_a^b f(x)\,dx < 0$ & $\int_c^b f(x)\,dx < 0$ but second term is large negative value so option (c) is incorrect.

(d) Clearly, (d) is incorrect.

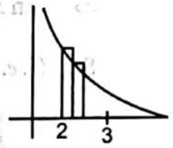


2. $T_n = \frac{1}{n}\sum_{r=2n}^{3n-1} \frac{(r/n)}{1+(r/n)^2}$, $S_n = \frac{1}{n}\sum_{r=2n+1}^{3n} \frac{(r/n)}{1+(r/n)^2}$

Let $f(x) = \frac{x}{1+x^2}$

$$f'(x) = \frac{(1+x^2)-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

$\therefore$ $f(x)$ is decreasing in (2, 3).

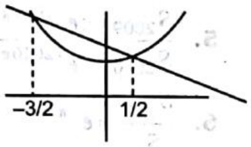


$$T_n > \int_2^3 f(x)\,dx, \quad S_n < \int_2^3 f(x)\,dx$$

3. $a+b=2$ ...(1)

$\int_0^4 (a\sqrt{x}+bx)\,dx = 8 \quad \Rightarrow \frac{2a}{3}+b=1$ ...(2)

4. Normal $y+x=\frac{7}{4}$



$$\int_{-3/2}^{1/2} \left(\frac{7}{4}-x\right)-(x^2+1)\,dx$$

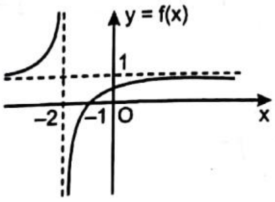
Exercise-3 : Comprehension Type Problems

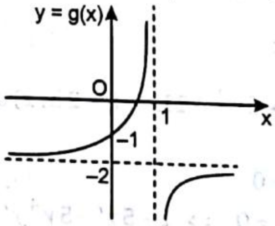
Paragraph for Question Nos. 1 to 3

Sol. $f(x) = \frac{x+a}{bx^2+cx+2}$

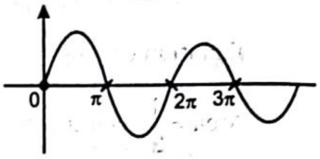
$f(-1)=0 \Rightarrow a=1$

If $y=1$ is asymptotes, then $b=0$ and $c=1 \Rightarrow f(x)=\frac{x+1}{x+2}$ and $g(x)=\frac{1-2x}{x-1}$





Paragraph for Question Nos. 4 to 6



Sol. $y = e^{-x}\sin x$

$$\frac{dy}{dx} = e^{-x}[\cos x - \sin x] = 0$$

$$\Rightarrow \tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}$$

So, $(i.e., 0 \le x \le \pi)$

$$I = \int e^{-x}\sin x\,dx = \frac{-e^{-x}}{2}[\sin x + \cos x] + c$$

$$S_j = \left|\int_{j\pi}^{(j+1)\pi} e^{-x}\sin x\,dx\right| = \left|-\frac{e^{-x}}{2}[\sin x + \cos x]\right|_{j\pi}^{(j+1)\pi} = \frac{e^{-j\pi}}{2}(e^{-\pi}+1)$$

4. Put $j = 0, S_0 = \frac{1+e^{-\pi}}{2}$

5. $\frac{S_{2009}}{S_{2010}} = \frac{e^{-2009\pi}}{e^{-2010\pi}} = e^{\pi}$

6. $\frac{S_{j+1}}{S_j} = e^{-\pi}$

$$\because \sum_{j=0}^{\infty} S_j = \frac{S_0}{1-e^{-\pi}} = \frac{\frac{1+e^{-\pi}}{2}}{1-e^{-\pi}} = \frac{1+e^{\pi}}{2(e^{\pi}-1)}$$

Exercise-5 : Subjective Type Problems

1. $f(x) = x^2$

$$A = 2\int_0^1 (\sqrt{2-x^2} - x^2)\,dx = \frac{\pi}{2} + \frac{1}{3}$$

2. $f(x) = 2\ln x$

$$A = \int_0^1 (-x^3 + 6x^2 - 11x + 6 - 2\ln x)\,dx = \frac{17}{4}$$

4. At $x = 0, y = 0$

$x + 5y - y^5 = 0 \Rightarrow 1 + 5y' - 5y^4y' = 0$

at $x = 0, y = 0$

$$y' = -\frac{1}{5}$$

Equation of tangent : $y = -\frac{x}{5}$, equation of normal : $y = 5x$

Area $= \frac{1}{2} \times 5 \times 26 = 65$

5. $[x]^2 = [y]^2$

$\therefore \quad [y] = \pm [x]$

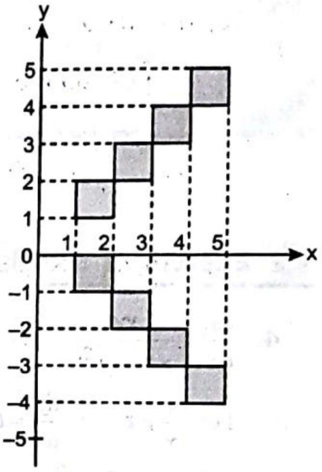
$[y] = \pm 1, \quad 1 \le x < 2$

$= \pm 2, \quad 2 \le x < 3$

$= \pm 3, \quad 3 \le x < 4$

$= \pm 4, \quad 4 \le x < 5$

$= \pm 5, \quad x = 5$



Now, when, $x \in [1, 2)$

then $y \in [-1, 0) \cup [1, 2)$

when $x \in [2, 3]$

then $y \in [-2, -1) \cup [2, 3)$

when $x \in [3, 4)$

then $y \in [-3, -2) \cup [3, 4)$

when $x \in [4, 5)$

then $y \in [-4, -3) \cup [4, 5)$

when $x = 5$

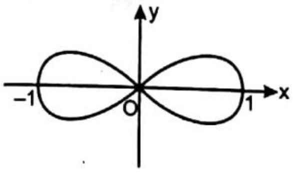
then $y \in [-5, -4) \cup [5, 6)$

Hence, required area $= 2(4) = 8$ sq. unit

6. $f(x) + f(z) = f(x + z)$ and $f(0) = 0$ and $f'(0) = 4 \Rightarrow f(x) = 4x$

So, area bounded $= \Delta = \int_0^4 (4x - x^2)\, dx$

7. Required area $= 4\int_0^1 \sqrt{x^2 - x^6}\, dx = \int_0^1 4x\sqrt{1 - x^4}\, dx = \frac{\pi}{2}$



Put $x^2 = \sin\theta$

8. Ar $= 4\int_0^1 (1 - x^{2/5})\, dx = 4\left(x - \frac{5}{7}x^{7/5}\right)_0^1 = \frac{8}{7}$

Chapter 7 – Differential Equations

7

DIFFERENTIAL EQUATIONS

Exercise-1 : Single Choice Problems

4. $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$

Let $\frac{y}{x} = t \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$

$\int \frac{dx}{x} = -\int \frac{1+t^2}{t^3}\,dt$; $\ln y = \frac{x^2}{2y^2} - \frac{1}{2}$ $(\because y(1) = 1)$

5. $\int \frac{y}{\sqrt{1-y^2}}\,dy = \int dx$

$-\sqrt{1-y^2} = x + c$

$\Rightarrow (x+c)^2 + y^2 = 1$

6. $\int \frac{dy}{y} = -\int \frac{dx}{(x-3)^2}$

$\Rightarrow \ln y = \frac{1}{x-3} + c$

7. Let $f(x) = y$

$\frac{dy}{dx} - 2xy = \frac{e^{x^2}}{(x+1)^2}$

8. Let $x^2y^2 = t$

$2xy^2 + 2x^2y\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} = \tan t$

9. $y=(C_1\cos C_2)\cos x+(C_5-C_1\sin C_2)\sin x-C_3e^{C_4}e^{-x}$

$y=A\cos x+B\sin x+Ce^{-x}$

$\because\quad C_1, C_2, C_3, C_4$ are arbitrary constants.

10. $y=e^{(\alpha+1)x}$

$y'=e^{(\alpha+1)x}(\alpha+1)$

$y''=e^{(\alpha+1)x}(\alpha+1)^2$

12. $\dfrac{dy}{dx}-\left(1+\dfrac{f'(x)}{f(x)}\right)y=f(x)$

$$\text{I.F.}=e^{-\int\left(1+\frac{f'(x)}{f(x)}\right)dx}=\frac{e^{-x}}{f(x)}$$

$$\frac{ye^{-x}}{f(x)}=\int e^{-x}dx+C \quad\Rightarrow\quad \frac{ye^{-x}}{f(x)}=-e^{-x}+C$$

13. Equation of tangent at $\left(t,\dfrac{t^2}{2}\right)$ is

$$y=tx-\frac{t^2}{2}\quad\Rightarrow t=\frac{dy}{dx}$$

Differential equation is

$$\left(\frac{dy}{dx}\right)^2-2x\frac{dy}{dx}+2y=0$$

14. Let $x+y=t\Rightarrow 1+\dfrac{dy}{dx}=\dfrac{dt}{dx};\quad \dfrac{1}{t^3}\dfrac{dt}{dx}+\dfrac{x}{t^2}=x\quad\Rightarrow\quad \dfrac{e^{-x^2}}{(x+y)^2}=e^{-x^2}+C$

15. $\dfrac{dy}{dx}-2y\tan x=\tan^2 x$

$$\text{I.F.}=e^{-2\int\tan x\,dx}=\cos^2 x$$

$$y\cos^2 x=\int\sin^2 x\,dx=\frac{1}{2}\int(1-\cos 2x)\,dx$$

16. $f(x)=2e^x+1$

17. Let $\dfrac{dy}{dx}=t$ then $\dfrac{d^2y}{dx^2}=\dfrac{dt}{dx}$

$$\frac{dt}{dx}=\frac{2tx}{x^2+1}$$

$$\Rightarrow\quad t=\frac{dy}{dx}=3(x^2+1)\qquad(\because y'(0)=3)$$

$\Rightarrow \quad y = x^3 + 3x + 1 \qquad (\because y(0) = 1)$

18. $cy^2 = 2x + c$

$2cyy' = 2 \Rightarrow c = \dfrac{1}{yy'}$

$y^2 = 2x\,yy' + 1$

19. Let $\operatorname{cosec} y = t$

$\Rightarrow \quad -\operatorname{cosec} y \cot y \, dy = dt$

$$\frac{dt}{dx} - \frac{t}{x} = -\frac{1}{x^2}$$

$$\Rightarrow \quad \frac{t}{x} = \frac{1}{2x^2} + c \quad \Rightarrow \quad \frac{1}{x \sin y} = \frac{1}{2x^2} + c$$

20. $\dfrac{xdy - ydx}{x^2} = \dfrac{\sqrt{x^2 + y^2}}{x^2}dx$

$$\int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1 + \left(\frac{y}{x}\right)^2}} = \int \frac{dx}{x}$$

21. $\lim\limits_{t \to x} \dfrac{t^3 f(x) - x^3 f(t)}{t^2 - x^2} = \dfrac{1}{2} \Rightarrow 3x\, f(x) - x^2 f'(x) = 1$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{3y}{x} = \frac{-1}{x^2} \quad \Rightarrow \quad y = \frac{1}{4x} + \frac{3}{4}x^2 \qquad (\because f(1) = 1)$$

22. $\dfrac{2dp(t)}{dt} = p(t) - 900$

$$2\int \frac{dp(t)}{p(t) - 900} = \int dt$$

$2\ln|900 - p(t)| = t + c$

$p(t) = 900 - 50e^{t/2} \qquad (\because p(0) = 850)$

$p(t) = 0 \Rightarrow t = 2\ln 18$

23. $\dfrac{\sin y}{\cos^2 y}\dfrac{dy}{dx} + \dfrac{\tan x}{\cos y} = \sec x$

Let $\dfrac{1}{\cos y} = t \Rightarrow \dfrac{\sin y}{\cos^2 y}dy = dt$

$$\frac{dt}{dx} + t \cdot \tan x = \sec x$$

$$t\cdot\sec x=\int\sec^2 x\cdot dx+C$$

$$\sec y\cdot\sec x=\tan x+C$$

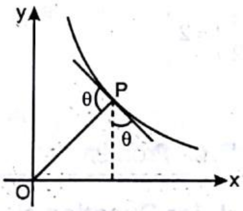
24. $\dfrac{dy}{dx}=(4x+y+1)^2$

Let $4x+y+1=t$

$$4+\frac{dy}{dx}=\frac{dt}{dx}\Rightarrow\frac{dt}{dx}=t^2+4$$

$$\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)=x+C$$

25. $\tan\theta=\left|\dfrac{\dfrac{dy}{dx}-\dfrac{y}{x}}{1+\dfrac{y}{x}\cdot\dfrac{dy}{dx}}\right|=-\dfrac{dx}{dy}$



$$\Rightarrow\left(\frac{dy}{dx}\right)^2-\frac{2y}{x}\frac{dy}{dx}=1$$

26. $\text{I.F.}=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$

$$y\cdot x=\int x^3dx=\frac{x^4}{4}+c$$

27. $x^3dy+3x^2y\,dx=y^2dx+2xy\,dy$

$$d(x^3y)=d(xy^2)$$

$$\int d(x^3y)=\int d(xy^2)\Rightarrow x^3y=xy^2$$

Exercise-2 : One or More than One Answer is/are Correct

1. $\dfrac{xdy-ydx}{x^2}=\dfrac{x^2-2}{x^2}dx$

$$\int d\left(\frac{y}{x}\right)=\int\left(1-\frac{2}{x^2}\right)dx$$

$y=x^2-2x+2$ $\qquad\qquad(\because f(1)=1)$

2. I.F. $= x\sec x$; $yx\sec x = \tan x + c$

3. Put $y = h$

$$\Rightarrow \quad x[f(x+h) - f(x-h)] - h[f(x+h) + f(x-h)] = 2(x^2h - h^3)$$

$$\text{or} \quad \lim_{h\to 0} x\frac{[f(x+h) - f(x-h)]}{h} - [f(x+h) + f(x-h)] = \lim_{h\to 0} 2(x^2 - h^2)$$

$$\Rightarrow \quad xf'(x) - f(x) = x^2 \quad \Rightarrow \quad f(x) = x^2 + x$$

4. L.D.E., I.F. $= 1 + \sin^2 x$; $(1 + \sin^2 x)f = \sin x + C$, $C = 0$

5. $2ydx + 2xdy + (2x^2y^{3/2}dx + x^3y^{1/2}dy) = 0$

$$2d(xy) + \frac{2}{3}d(x^3 \times y^{3/2}) = 0$$

6. I.F. $= \dfrac{1}{\sin^3 x}$; $\dfrac{y}{\sin^3 x} = \displaystyle\int \frac{\sin 2x}{\sin^3 x}dx$; $\dfrac{y}{\sin^3 x} = 2\displaystyle\int \cot x \cdot \text{cosec}\, x\, dx = -2\,\text{cosec}\, x + c$

$$y = -2\sin^2 x + 4\sin^3 x \quad \left(\because y\left(\frac{\pi}{2}\right) = 2\right)$$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

Sol. $x\displaystyle\int_0^x g(t)\,dt + \int_0^x (1-t)g(t)\,dt = x^4 + x^2$

differentiate w.r.t. 'x'

$$x\,g(x) + \int_0^x g(t)\,dt + (1-x)\,g(x) = 4x^3 + 2x \qquad \text{...(1)}$$

1. From (1)

$$\int_0^x g(t)\,dt + g(x) = 4x^3 + 2x$$

Let $g(x) + g'(x) = 12x^2 + 2 \;\Rightarrow\; \dfrac{dy}{dx} + y = 12x^2 + 2$ $\qquad (\because y = g(x))$

2. Put $x = 0$ in (1) we get $g(0) = 0$

Paragraph for Question Nos. 3 to 5

3. $f(g(x)) = e^{-2x}$

$$\frac{x \cdot [f(g(x))]'}{f(g(x))} = \frac{[g(f(x))]'}{g[f(x)]}$$

$$\Rightarrow \quad g(f(x)) = e^{-x^2}$$

$$H(x) = e^{-(x-1)^2+1}$$

4. $f(g(0)) + g(f(0)) = 2$

5. $H(x)_{\max} = e$

Paragraph for Question Nos. 6 to 8

Sol. $g'(x) = \lim\limits_{h\to 0} \dfrac{g(x+h) - g(x)}{h} = \lim\limits_{h\to 0} \dfrac{e^x g(h)}{h} + \lim\limits_{h\to 0} g(x)\left(\dfrac{e^h - 1}{h}\right)$ $\quad (\because g'(0) = 2)$

$$= 2e^x + g(x)$$

$$\frac{dy}{dx} - y = 2e^x \Rightarrow y = 2xe^x + ce^x$$

$$\Rightarrow \quad y = 2xe^x \qquad (\because g(0) = 0)$$

Exercise-4 : Matching Type Problems

1. (A) $y\dfrac{dx}{dy} - x = y^2\dfrac{dx}{dy} + 1 \Rightarrow (y^2 - y)\dfrac{dx}{dy} = -(1+x) \Rightarrow \dfrac{dx}{1+x} = -\dfrac{dy}{y(y-1)}$

(B) $y\dfrac{dx}{dy} + 2x = 10y^3 \Rightarrow \dfrac{dx}{dy} + \left(\dfrac{2}{y}\right)x = 10y^2$

I.E. $= e^{\int \frac{2}{y}dy} = e^{2\ln y} = y^2$

$d(xy^2) = 10y^4$

(C) $\dfrac{dy}{dx} = y'$

$y'y''' = (3y'')^2$

$\dfrac{y'''}{y''} = \dfrac{3y''}{y'}$ then integrate it.

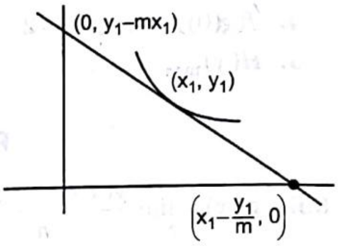
(D) Put $x^2 = t$

$\dfrac{dt}{dy} + \dfrac{t}{y} = \dfrac{1}{y^3}$

then solve it.

Exercise-5 : Subjective Type Problems

1. $x^a \cdot y = \lambda^a$; $\quad y = \dfrac{\lambda^a}{x^a}$



$$\frac{dy}{dx} = -a\lambda^a x^{-a-1} = -a\frac{x^a \cdot y}{x^{a+1}}$$

$$\Rightarrow \quad m = \frac{-ay_1}{x_1}$$

$$A = \frac{1}{2}|y_1 - mx_1|\left|x_1 - \frac{y_1}{m}\right| = \frac{1}{2}y_1x_1(1+a)^2$$

$$= \frac{1}{2}\lambda^a \cdot x_1^{1-a}(1+a)^2$$

For A to be constant $1 - a = 0$.

2. $\dfrac{dy}{dx} = xy(1+y)$

$$\int \frac{dy}{(1+y)\,y} = \int x\,dx$$

$$\frac{2y}{1+y} = e^{\frac{x^2}{2}} \qquad (\because f(0) = 1)$$

$$\Rightarrow \quad f(2) = \frac{e^2}{2 - e^2}$$

3. $y^2 = \cos^2 x + 2$

$$2y\frac{dy}{dx} = -\sin 2x$$

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\cos 2x$$

$$y^4 + y^3\frac{d^2y}{dx^2} = (\cos^2 x + 2)^2 + (\cos^2 x + 2)\left[-\left(\frac{dy}{dx}\right)^2 - \cos 2x\right] = 6$$

4. $\displaystyle\lim_{t \to x+1} \frac{t^2 f(x+1) - (x+1)^2 f(t)}{f(t) - f(x+1)} = 1$

$$\Rightarrow \quad \lim_{t \to x+1} \frac{2tf(x+1) - (x+1)^2 f'(t)}{f'(t)} = 1$$

$$\Rightarrow \quad [x+1][2f(x+1) - (x+1)f'(x+1)] = f'(x+1)$$

$$\Rightarrow \quad f'(x) = \frac{2xf(x)}{x^2+1}$$

$$\Rightarrow \quad f(x) = x^2 + 1$$

$$\Rightarrow \quad \lim_{x\to 1} \frac{\ln(f(x)) - \ln 2}{x-1} = \lim_{x\to 1} \frac{f'(x)}{f(x)} = 1$$

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Chapter 8 – Quadratic Equations

8

QUADRATIC EQUATIONS

Exercise-1 : Single Choice Problems

1. Let $3^{x/2}=a, 2^y=b$

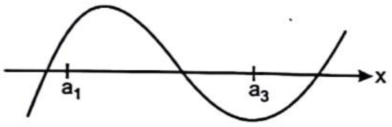
$a^2-b^2=77, a-b=7 \Rightarrow a=3^{x/2}=9 \Rightarrow x=4$

$b=2^y=2 \Rightarrow y=1$

2. $f(x)=\prod_{i=1}^{3}(x-a_i)+\sum_{i=1}^{3}a_i-3x=(x-a_1)(x-a_2)(x-a_3)+(a_1+a_2+a_3)-3x$

$f(a_1)=a_2+a_3-2a_1>0 \qquad (a_1<a_2<a_3)$

$f(a_3)=a_1+a_2-2a_3<0$



3. $x^4-2ax^2+x+a^2-a=0$

$a^2-a(2x^2+1)+x^4+x=0$

$$a=\frac{2x^2+1\pm(2x-1)}{2}$$

$a=x^2+x, \quad a=x^2-x+1$

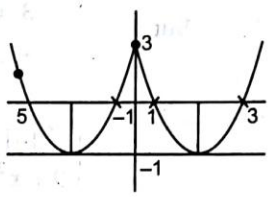
$a\geq-\frac{1}{4}, \quad a\geq\frac{3}{4} \qquad (\because x\in R)$

4. $x^3-3x^2-4x+12=0 \begin{array}{l}\nearrow\alpha\\ \rightarrow\beta\\ \searrow\gamma\end{array}$

Equation whose roots are $\alpha-3, \beta-3, \gamma-3$ is

$(x+3)^3-3(x+3)^2-4(x+3)+12=0$

$$f(x)=x^3+6x^2+5x=0 \begin{array}{l}\nearrow\alpha-3\\ \rightarrow\beta-3\\ \searrow\gamma-3\end{array}$$

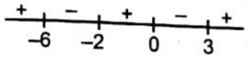


5. $|K-1|<3$

$-3<K-1<3$

$-2<K<4$

6. $\dfrac{x}{x+6}-\dfrac{1}{x}\leq 0 \Rightarrow \dfrac{x^2-x-6}{x(x+6)}\leq 0 \Rightarrow \dfrac{(x-3)(x+2)}{x(x+6)}\leq 0$



$x\in(-6,-2]\cup(0,3]$

7. $P(x)=x^4-8x^2+15+2x^3-6x=(x^2-3)(x^2-5)+2x(x^2-3)$

$$=(x^2-3)(x^2+2x-5)$$

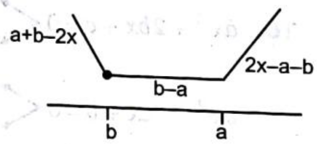
$Q(x)=(x+2)(x^2+2x-5)$

8. $a=1, h=\dfrac{\lambda}{2}, b=1, g=\dfrac{-5}{2}, f=\dfrac{-7}{2}, c=6$

$$\begin{vmatrix} 1 & \lambda/2 & -5/2 \\ \lambda/2 & 1 & -7/2 \\ -5/2 & -7/2 & 6 \end{vmatrix}=0 \Rightarrow \lambda=\frac{5}{2},\frac{10}{3}$$

10. $\therefore$ Let $f(x)=|x-a|+|x-b|$

Suppose $a>b$



$\therefore \quad f(0)=f(1)=f(-1)$

$f(x)=\text{const. in }[b,a]$

So, $\quad b\leq -1<a\leq 1$

$a-b\leq 2$

$\therefore \quad \text{Minimum}\,|a-b|=2$

12. $y=\dfrac{x^2+2x+c}{x^2+4x+3c} \Rightarrow (y-1)x^2+2(2y-1)x+(3cy-c)=0 \qquad (D\geq 0)$

$D\geq 0\ \forall\ y\in R$ and $D\leq 0$

But at $c=0$ and 1 there will be common factors among numerator and denominator.

$\Rightarrow \quad c(c-1)<0$

13. $f(t)=t^2-mt+2=0$

$f(2)<0$

$\Rightarrow \quad 4-2m+2<0 \Rightarrow m>3$

But $$\frac{3|x|}{9+|x|^2}=\frac{3}{\frac{9}{|x|}+|x|}\le\frac{3}{6}$$ (by A.M. G.M. in equality)

$$\left(\frac{3|x|}{9+x^2}\right)^m\le\frac{1}{2^m}<1 \qquad [\because m>3]$$

So, $$\left[\left(\frac{3|x|}{9+x^2}\right)^m\right]=0$$

14. $x^2(x^6-24x^5-18x^3+39)=-3\times5\times7\times11$

If 'x' is integer, then there is no value of 'x'.

15. $$m^4+\frac{1}{m^4}=119$$

$$\Rightarrow\ m^2+\frac{1}{m^2}=11$$

$$\Rightarrow\ \left(m-\frac{1}{m}\right)^2=9$$

$$\left|m^3-\frac{1}{m^3}\right|=\left|\left(m-\frac{1}{m}\right)\left(m^2+\frac{1}{m^2}+1\right)\right|=|3\times12|=36$$

16. $ax^2+2bx+c=0 \begin{array}{l}\nearrow\alpha\\ \searrow\beta\end{array}$

$ax^2+2cx+b=0 \begin{array}{l}\nearrow\alpha\\ \searrow\gamma\end{array}$

By condition of common root

$\Rightarrow$ Common root $\alpha=\frac{1}{2}$ and $\frac{a}{4}+b+c=0$

$\Rightarrow$ $\beta=\frac{2c}{a}$ and $\gamma=\frac{2b}{a}$

Equation whose roots are β and γ is

$$x^2-\left(\frac{2c}{a}+\frac{2b}{a}\right)x+\frac{4bc}{a^2}=0$$

$$2a^2x^2+a^2x+8bc=0 \qquad \left(\frac{a}{4}+b+c=0\right)$$

17. $$9x^2(x-1)-1(x-1)=0$$

$$x=1,\ x=\frac{1}{3},-\frac{1}{3}$$

$$\cos\alpha=1,\ \cos\beta=\frac{1}{3},\ \cos\gamma=-\frac{1}{3}$$

$$\alpha = 0, \beta + \gamma = \pi$$

$\therefore$ $(\Sigma\alpha, \Sigma\cos\alpha) = (\pi, 1) = \text{centre}$

$$\left[2\sin^{-1}\left(\tan\frac{\pi}{4}\right), 4\right] = \left[2\left(\frac{\pi}{2}\right), 4\right] = (\pi, 4) \rightarrow \text{point lies on the circle.}$$

$\therefore$ Radius is 3.

18. $y = \dfrac{11x^2 - 12x - 6}{x^2 + 4x + 2}$

$(y - 11)x^2 + (4y + 12)x + (2y + 6) = 0 \ \forall\ x \in R$

$$D \geq 0$$

$(4y + 12)^2 - 4(y - 11)(2y + 6) \geq 0$

$$y^2 + 20y + 51 \geq 0$$

$$(y + 17)(y + 3) \geq 0$$

$$y \in (-\infty, -17] \cup [-3, \infty)$$

19. $\dfrac{x + 3}{x^2 - x - 2} - \dfrac{1}{x - 4} \geq 0$

$$\frac{(x^2 - x - 12) - (x^2 - x - 2)}{(x^2 - x - 2)(x - 4)} \geq 0$$

$$\frac{-10}{(x - 2)(x + 1)(x - 4)} \geq 0$$

$\Rightarrow$ $(x + 1)(x - 2)(x - 4) < 0$

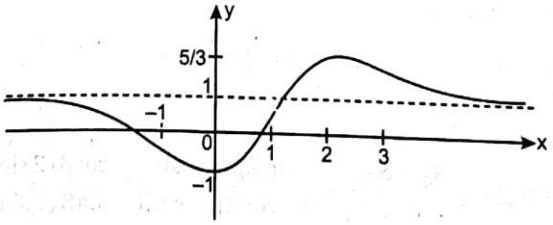
$$x \in (-\infty, -1) \cup (2, 4)$$

20. $x = 4 + 3i$

$(x - 4)^2 = -9 \ \Rightarrow\ x^2 - 8x + 25 = 0$

$x^3 - 4x^2 - 7x + 12 = (x^2 - 8x + 25)(x + 4) - 88 = -88$

21. $f(x) = \dfrac{x^2 + x - 1}{x^2 - x + 1}$



By graph : Min. = $f(0)$; Max. = $f(2)$

If $x \in [-1, 3]$; $y_{\text{max.}} = \dfrac{5}{3}$

22. By graph min. = $f(0)$; max. = $f(1)$
if $x \in [-1, 1]$; $y \in [-1, 1]$

23.
$$\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$$

$x^2 + x(p+q-2r) + pq - r(p+q) = 0$

If one root is α. Then other root must be $-\alpha$.

$$p + q - 2r = 0 \;\Rightarrow\; r = \frac{p+q}{2}$$

Product of the roots $= pq - r(p+q) = pq - \dfrac{(p+q)^2}{2} = -\dfrac{(p^2+q^2)}{2}$

24. If $a_1x^2 + b_1x + c_1 = 0$ has one root α.

$\Rightarrow$ $a_2x^2 + b_2x + c_2 = 0$ has one root $\dfrac{1}{\alpha}$.

$\Rightarrow$ $c_2x^2 + b_2x + a_2 = 0$ has one root α

Condition of common root is

$$(a_1a_2 - c_1c_2)^2 = (a_1b_2 - b_1c_2)(a_2b_1 - b_2c_1)$$

25. If $\alpha^2 - 5\alpha + 3 = 0$ and $\beta^2 - 5\beta + 3 = 0$

$\Rightarrow$ $x^2 - 5x + 3 = 0$ has two roots α and β.

$\Rightarrow$ $\alpha + \beta = 5,\ \alpha\beta = 3$

Sum of the roots $= \dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \dfrac{19}{3}$

Product of roots $= \dfrac{\alpha}{\beta} \times \dfrac{\beta}{\alpha} = 1$

Equation whose roots are $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ is $3x^2 - 19x + 3 = 0$

26. $|\alpha - \beta| = |\alpha_1 - \beta_1|$

$a^2 - 4b = b^2 - 4a$

$a^2 - b^2 = 4(b - a)$

$(a-b)(a+b+4) = 0$

$a \neq b \Rightarrow a + b + 4 = 0$

27. $\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \dfrac{S_1 - S_3}{1 - S_2 + S_4} = \dfrac{\sin 2\beta - \cos\beta}{1 - \cos 2\beta - \sin\beta} = \dfrac{\cos\beta\,(2\sin\beta - 1)}{\sin\beta\,(2\sin\beta - 1)} = \cot\beta$

28. $(a^2 + b^2)x^2 + 2x(bd + ac) + (c^2 + d^2) = 0$

$(a^2x^2 + 2acx + c^2) + (b^2x^2 + 2bdx + d^2) = 0$

$$(ax+c)^2+(bx+d)^2>0$$

$\Rightarrow$ This equation has imaginary roots.

29. If α,β are roots of $ax^2+bx+c=0$

$\Rightarrow$ $\alpha+2,\beta+2$ are roots of $a(x-2)^2+b(x-2)+c=0$

$\Rightarrow$ $ax^2+x(b-4a)+4a-2b+c=0$

30.
$$\alpha+\beta=1+\lambda$$
$$\alpha\beta=\lambda-2$$
$$\alpha+\beta-\alpha\beta=3$$
$$(\alpha-1)(\beta-1)=-2$$

$\Rightarrow$ atleast one root is positive.

31. $D\geq 0\Rightarrow 3k^2+8k-16\leq 0\Rightarrow -4\leq k\leq\frac{4}{3}$

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=k^2-2(k^2+2k-4)=-k^2-4k+8=12-(k+2)^2$$

32.
$$P(x)=(x-2)Q_1(x)+R(x)$$
$$Q(x)=(x-2)Q_2(x)+R(x)$$

$\Rightarrow$ $P(2)=Q(2)$

33. $a+b=-a$ and $ab=b$

if $b\neq 0, a=1$ and $b=-2$

$$x^2+ax+b=x^2+x-2=\left(x+\frac{1}{2}\right)^2-\frac{9}{4}$$

34. $x^2+\left(\frac{b}{a}\right)\left(\frac{c}{a}\right)x+\left(\frac{c}{a}\right)^3=0$

$$x^2-(\alpha+\beta)\cdot\alpha\beta\,x+\alpha^3\beta^3=0\begin{array}{l}\nearrow\alpha^2\beta\\ \searrow\alpha\beta^2\end{array}$$

35. $x^2+2(a+b+c)x+6k(ab+bc+ca)=0$

$D\geq 0$

$\Rightarrow$ $4(a+b+c)^2-24k(ab+bc+ca)\geq 0$

$\Rightarrow$ $k\leq\frac{1}{6}\left(\frac{a^2+b^2+c^2}{ab+bc+ca}+2\right)$

also, $|a-b|<c,\ |b-c|<a,\ |c-a|<b$

$\Rightarrow$ $a^2+b^2+c^2-2(ab+bc+ca)<0$

$\Rightarrow$ $\frac{a^2+b^2+c^2}{ab+bc+ca}<2\ \Rightarrow\ k<\frac{2}{3}$

36. $$9|x|^2-18|x|+5=0$$

$\Rightarrow \quad (3|x|-1)(3|x|-5)=0$

$$\Rightarrow \qquad x=\pm\frac{1}{3},\pm\frac{5}{3}$$

and $x^2-x-2>0 \;\Rightarrow\; (x-2)(x+1)>0 \;\Rightarrow\; x<-1$ or $x>2$

37. Difference of roots is same in both equation

$$b^2-c=B^2-C$$

38. $|x-p|+|x-15|+|x-p-15|=(x-p)-(x-15)-(x-P-15)=30-x$

$$\text{min.}=15$$

39. $$4p(q-r)x^2-2q(r-p)x+r(p-q)=0 \begin{cases} \rightarrow \alpha=-1/2 \\ \rightarrow \beta=-1/2 \end{cases}$$

If $x=-\frac{1}{2}$ is also the root of $4x^2-2x-m=0$

$\Rightarrow \quad m=2$

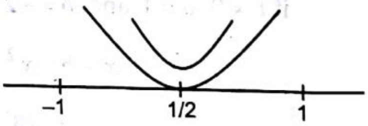
40. Let $\cos x=t$

$\Rightarrow \qquad t\in[-1,1]$

$\Rightarrow \qquad kt^2-kt+1\geq 0 \;\forall\; t\in[-1,1]$

Case I : $k\geq 0$

x coordinate of vertex is $\frac{1}{2}$.



$\Rightarrow \qquad f\left(\frac{1}{2}\right)\geq 0$

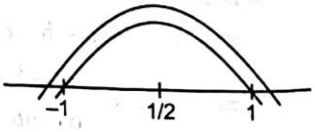
$\Rightarrow \qquad \frac{k}{4}-\frac{k}{2}+1\geq 0$

$\Rightarrow \qquad k\leq 4$

Also, $\qquad k\geq 0$

$\Rightarrow \qquad k\in[0,4]$

Case II : $k<0$



$\Rightarrow \qquad f(1)\geq 0 \quad$ and $\quad f(-1)\geq 0$

$\qquad k-k+1\geq 0 \quad$ and $\quad k+k+1\geq 0$

$\Rightarrow \qquad 1\geq 0 \quad$ and $\quad k\geq -\frac{1}{2}$

Also, $\qquad k<0$

$\Rightarrow \qquad k\in\left[-\frac{1}{2},0\right) \quad \Rightarrow \quad k\in\left[-\frac{1}{2},4\right]$

41. $\frac{1+x}{1-x}=y \Rightarrow x=\frac{y-1}{y+1}$

$H(y)=3\left(\frac{y-1}{y+1}\right)^3-2\left(\frac{y-1}{y+1}\right)+5=0$

$H(y)=3(y-1)^3-2(y-1)(y+1)^2+5(y+1)^3=0$

$H(y)=3(y^3-3y^2+3y-1)-2(y-1)(y^2+2y+1)+5(y^3+3y^2+3y+1)=0$

$H(y)=3(y^3-3y^2+3y-1)-2(y^3+y^2-y-1)+5(y^3+3y^2+3y+1)=0$

$H(y)=3y^3+2y^2+13y+2=0$

$H'(x)=9x^2+4x+13 \Rightarrow D<0$

$H(x)>0 \;\forall\; x>0$

Hence, it has one –ve real root.

42. $(\lambda^2+\lambda-2)x^2+(\lambda+2)x-1<0 \;\forall\; x\in R$

$\lambda^2+\lambda-2<0 \cap (\lambda+2)^2+4(\lambda^2+\lambda-2)<0$

$(\lambda+2)(\lambda-1)<0 \cap 5\lambda^2+8\lambda-4<0$

$\lambda\in(-2,1)\cap\lambda\in\left(-2,\frac{2}{5}\right)$

$\Rightarrow \quad \lambda\in\left(-2,\frac{2}{5}\right)$

$\lambda=-2$ is also the solution of this equation.

43. $\alpha=1,\beta=1,\gamma=1,\delta=1$ (as) $(\alpha-1)^2+(\beta-1)^2+(\gamma-1)^2+(\delta-1)^2=0$

$\therefore$ The roots of given equation is equal to 1.

$\therefore \quad S_2=\frac{a_2}{a_0}=6$

44. $|x-1|+|x-2|+|x-3|\ge 6$

Case I : $x\ge 3$

$3x-6\ge 6 \Rightarrow x\ge 4$

Case II : $2<x<3$

$x\ge 6$ (Not possible)

Case III : $1\le x\le 2$

$4-x\ge 6$

$\Rightarrow$ $x\le -2$ (Not possible)

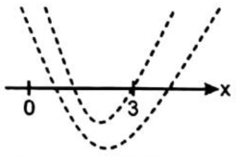
Case IV : $x<1$

$6-3x\ge 6$

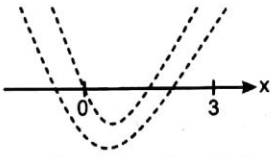
$x\le 0$

$x\in(-\infty,0]\cup[4,\infty)$

45.



Case-I : $f(0) > 0 \cap f(3) \le 0$



Case-II : $f(3) > 0 \cap f(0) \le 0$

46. $x^3 + 3px^2 + 3qx + r = 0 \begin{array}{l}\nearrow \alpha \\ \rightarrow \beta \\ \searrow \gamma\end{array}$

$$\frac{2}{\beta} = \frac{1}{\alpha} + \frac{1}{\gamma} \ldots\ldots \qquad (\because \alpha, \beta, \gamma \text{ are in H.P.})$$

$$\Rightarrow \qquad \frac{3}{\beta} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma}$$

$$\Rightarrow \qquad \beta = -\frac{r}{q} \qquad \text{which satisfy the given equation.}$$

47. $4y^2 + 4xy + (x+6) = 0 \; \forall \; y \in R$

$D \ge 0 \;\Rightarrow\; x^2 - x - 6 \ge 0$

48. $\log_{\cos x^2}(3 - 2x) < \log_{\cos x^2}(2x - 1)$

$0 < \cos x^2 < 1 \cap 3 - 2x > 2x - 1 \cap 3 - 2x > 0 \cap 2x - 1 > 0$

$x < 1 \qquad\qquad x < 3/2 \qquad x > 1/2$

49. $px^2 + qx + r = 0 \begin{array}{l}\nearrow \alpha \\ \searrow \beta\end{array}$

$\Rightarrow \qquad \alpha\beta < 0$

$\alpha(x - \beta)^2 + \beta(x - \alpha)^2 = (\alpha + \beta)x^2 - 4\alpha\beta x + \alpha\beta(\alpha + \beta) = 0$

Product of roots $= \alpha\beta < 0$

$D = 16\alpha^2\beta^2 - 4\alpha\beta(\alpha + \beta)^2 = -4\alpha\beta(\alpha - \beta)^2 > 0$

50. $x^3 + 2x^2 - 4x - 4 = 0 \begin{array}{l}\nearrow a \\ \rightarrow b \\ \searrow c\end{array}$

$4x^3 + 4x^2 - 2x - 1 = 0 \begin{array}{l}\nearrow 1/a \\ \rightarrow 1/b \\ \searrow 1/c\end{array}$

$q = 1, r = -\frac{1}{2}, s = -\frac{1}{4}$

51. $\log_2(x^2 + 3x) \le 2$

$0 < x^2 + 3x \le 4$

52. $k-2>0 \cap D<0$

$k>2 \cap (k+6)(k-4)>0$

$\Rightarrow \quad k>4$

53. $\alpha\beta<0$

$\dfrac{3m-8}{m-2}<0 \;\Rightarrow\; 2<m<\dfrac{8}{3}$

54. $\log_6\left(\dfrac{x^2+x}{x+4}\right)>1$

$\dfrac{x^2+x}{x+4}>6 \;\Rightarrow\; \dfrac{x^2-5x-24}{x+4}>0 \;\Rightarrow\; \dfrac{(x-8)(x+3)}{x+4}>0$

55. $ax^2+c=0 \begin{array}{l} \nearrow \alpha \\ \searrow \beta \end{array}$

$\alpha+\beta=0,\ \alpha\beta=\dfrac{c}{a}$

$\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)=0$

56. $(k-1)x^2-(k+1)x+(k+1)>0\ \forall\ x\in R$

$k-1>0 \cap \ (k+1)^2-4(k-1)(k+1)<0$

$k>1 \quad \cap \ (k+1)(3k-5)>0$

$\Rightarrow \quad k>\dfrac{5}{3}$

57. $y=-2x^2-4ax+k$; abscissa corresponding to the vertex is $-\dfrac{b}{2a}$ $i.e.,\left(\dfrac{4a}{-4}\right)=-2 \Rightarrow a=2$

now, $\quad y(-2)=7$

$$7=-8+16+k \Rightarrow k=-1$$

58. If $a+b+c=0$

Sum of coefficient $(b+c-a)+(c+a-b)+(a+b-c)=a+b+c=0$

$\Rightarrow$ $x=1$ is one root of the equation.

$\Rightarrow$ other root $=\dfrac{a+b-c}{b+c-a}$

59. $x^3-ax^2+bx-c=0 \begin{array}{l} \nearrow \alpha \\ \rightarrow -\alpha \\ \searrow \beta \end{array}$

Sum of roots $\alpha-\alpha+\beta=a \Rightarrow \beta=a$

If β is root of the equation, then $ab=c$.

60. $\alpha'\beta'=2q^2-r=2\alpha^2\beta^2-(\alpha^4+\beta^4)=-(\alpha^2-\beta^2)^2<0$

62. In ΔABC,

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cdot\cot\frac{C}{2}$$

If $\cot\frac{A}{2},\cot\frac{B}{2},\cot\frac{C}{2}$ are in A.P. then $\cot\frac{A}{2}+\cot\frac{C}{2}=2\cot\frac{B}{2}$

$$\Rightarrow \qquad \cot\frac{A}{2}\cot\frac{C}{2}=3$$

63. $f(x)=x\,\forall\, x\in[-9,9]$

64.
$$(3|x|-3)^2=|x|+7$$

$\Rightarrow\ (|x|-2)(9|x|-1)=0$

$$|x|=2,\frac{1}{9}\Rightarrow x=\pm2,\pm\frac{1}{9}$$

$$y=\sqrt{x(x-4)}$$

$D_f:(-\infty,0]\cup[4,\infty)$

65. $x^2+3|x|+2=0\Rightarrow(|x|+2)(|x|+1)=0$

66. $x^2-bx+c=0 \begin{cases}\alpha\\ \alpha+1\end{cases}$

Sum of roots $2\alpha=b-1\Rightarrow\alpha=\dfrac{b-1}{2}$

If α is the root of equation, then $\left(\dfrac{b-1}{2}\right)^2-b\left(\dfrac{b-1}{2}\right)+c=0\Rightarrow b^2-4c=1$

67. $y=\dfrac{3x^2+9x+17}{3x^2+9x+7}$

$$3(y-1)x^2+9(y-1)x+(7y-17)=0$$

$y-1\neq0$ then $D\geq0$

$$81(y-1)^2-12(y-1)(7y-17)\geq0$$

$$(y-1)(y-41)\leq0$$

$$1<y\leq41$$

68. $\dfrac{x^2+2x+7}{2x+3}-6<0\,\forall\, x\in R$

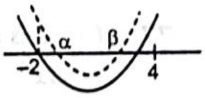
$$\frac{x^2-10x-11}{(2x+3)}<0$$

$$\frac{(x-11)(x+1)}{2x+3}<0$$

$$x\in\left(-\infty,-\frac{3}{2}\right)\cup(-1,11)$$

69. $y=\dfrac{3x-2}{7x+5} \Rightarrow x=\dfrac{5y+2}{3-7y} \Rightarrow y \in R-\left\{\dfrac{3}{7}\right\}$

70. $\dfrac{x+2}{x-4} \le 0 \Rightarrow x \in [-2,4)$



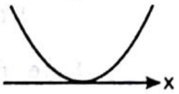
$x^2-ax-4 \le 0$

$f(-2) \ge 0 \cap f(4) > 0$

$a \ge 0 \cap a < 3$

$\Rightarrow \quad a \in [0,3)$

71. $P(x)=(P-3)x^2-2Px+(3P-6) \; \forall \; x \in R$



$P-3>0 \quad \cap \quad D=0$

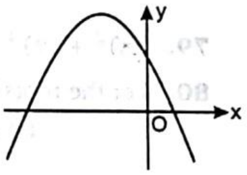
$P>3 \quad \cap \quad P^2-(P-3)(3P-6)=0$

$\Rightarrow \quad P=6$

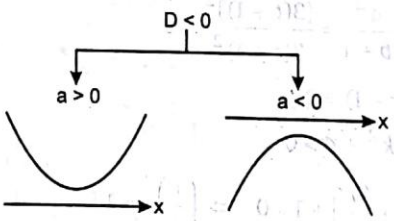
72. Graph is downward $\Rightarrow a<0$

Graph cut y-axis $\Rightarrow c>0$

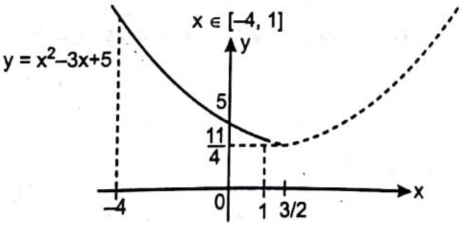
x-coordinate of vertex $\dfrac{-b}{2a}<0 \Rightarrow b<0$



73. $ax^2+bx+c=0$ does not have real roots.



74.



$y \in [3,33]$

75. $3x^2-17x+10=0 \Rightarrow (x-5)(3x-2)=0$

If $x=5$ is common root, then $m=0$

If $x=\dfrac{2}{3}$ is common root, then $m=\dfrac{26}{9}$

76. $x^2+(y+2)x-(y^2+y-1)=0$

$D \geq 0 \Rightarrow (y+2)^2+4(y^2+y-1) \geq 0 \Rightarrow y \in \left(-\infty, -\frac{8}{5}\right] \cup [0, \infty)$

77. If $x=3$ is root of this equation, then $k=-5$

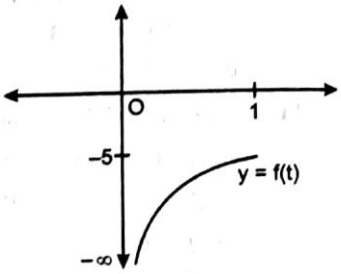
$\Rightarrow \quad 3x^4-6x^3-5x^2-8x-12=(x-3)(3x^2+4)(x+1)$

78. $a=-\frac{(4+\sin^4 x)}{\sin^2 x}$ put $\sin^2 x = t \Rightarrow t \in [0, 1]$

$a=-\left(\frac{4}{t}+t\right)=f(t)$

Here, $f'(t)=\frac{4}{t^2}-1>0$

$\therefore$ For atleast one real root, $a \in (-\infty, -5]$



79. $(rs)^2+(st)^2+(tr)^2=(rs+st+tr)^2-2rst(r+s+t)=b^2-2(-c)(-a)$

80. Let the roots be $t, t+1$ and $t+2$.

$$t+(t+1)+(t+2)=-a \quad \Rightarrow 3(t+1)=-a$$

$$\sum t(t+1)=b \quad \Rightarrow \quad b+1=3(t+1)^2$$

$$\frac{a^2}{b+1}=\frac{[3(t+1)]^2}{3(t+1)^2}=3$$

81. $(3x^2+kx+3)(x^2+kx-1)=0$

$D_1=k^2-36$ and $D_2=k^2+4>0$

82. $\frac{1}{r+s}=\frac{1}{r}+\frac{1}{s} \Rightarrow \left(\frac{r}{s}\right)^2+\left(\frac{r}{s}\right)+1=0 \Rightarrow \left(\frac{r}{s}\right)^3=1$

84. If $x \in (-\infty, -2] \cup [3, \infty)$

$x^2-2x-8=0 \Rightarrow x=-2, 4$

if $x \in (-2, 3)$

$x^2=4 \Rightarrow x=\pm 2$

85. $5x^2+12x+3=0$ has $D<0$

$\Rightarrow$ Both roots common.

86.

$$\alpha+\beta+\gamma=6$$

$$\alpha\beta+\beta\gamma+\alpha\gamma=5$$

$$\alpha\beta\gamma=1$$

$\Rightarrow \quad \alpha^2+\beta^2+\gamma^2=26$

$\alpha^2\beta^2+\beta^2\gamma^2+\alpha^2\gamma^2=13$

87. $2x^2-6x+k=0 \begin{array}{l} \nearrow \dfrac{\alpha+5i}{2} \\ \searrow \dfrac{\alpha-5i}{2} \end{array}$

Sum of roots $=\alpha=3$

Product of roots $=\dfrac{\alpha^2+25}{4}=\dfrac{k}{2} \Rightarrow k=17$

88. $x_1^2+x_2^2=(k-2)^2-2(k^2+3k+5)=-(k^2+10k+6)\le 18$

89. $a(x^2-x+1)-(x^2+x+1)\ge 0$

$$\Rightarrow \qquad a\ge \frac{x^2+x+1}{x^2-x+1}$$

90. $f(1)=\lambda-13>0 \;\Rightarrow \lambda>13$

$f(2)=\lambda-18<0 \;\Rightarrow \lambda<18$

$f(3)=\lambda-15>0 \;\Rightarrow \lambda>15$

$\Rightarrow \lambda\in(15,18)$

94. $D=(b-c)^2+4a(2b+a+c)=(b-c)^2+(4ac-4b^2)+(2a+2b)^2>0$

95. $x^3-x+1=0 \begin{array}{l} \nearrow a \\ \rightarrow b \\ \searrow c \end{array}$

$$(1-x)^3-x^2(1-x)+x^3=0 \begin{array}{l} \nearrow \dfrac{1}{a+1} \\ \rightarrow \dfrac{1}{b+1} \\ \searrow \dfrac{1}{c+1} \end{array}$$

$$\Rightarrow \qquad x^3+2x^2-3x+1=0$$

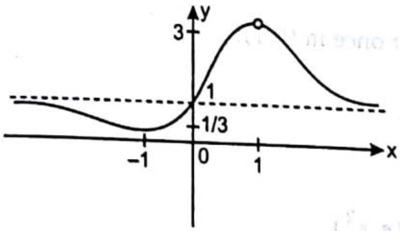
Sum of roots $=\dfrac{1}{1+a}+\dfrac{1}{1+b}+\dfrac{1}{1+c}=-2$

96. $x^2-2(4k-1)x+15k^2-2k-7\ge 0\;\forall\, x\in R$

$D\le 0$

$\Rightarrow\; k^2-6k-8\le 0 \;\Rightarrow\; 2\le k\le 4$

97. $f(x)=\dfrac{x^3-1}{(x-1)(x^2-x+1)}=\dfrac{x^2+x+1}{x^2-x+1} \qquad (\because x\ne 1)$



98. $\frac{2x^2+2}{x^2+mx+4}>0\ \forall\ x\in R$ $\quad\Rightarrow x^2+mx+4>0\ \forall\ x\in R$

$\Rightarrow D<0\Rightarrow m^2-16<0$

99. $x^2-2|a+1|x+1=0$

$D\geq 0\Rightarrow 4(a+1)^2-4\geq 0\Rightarrow a\in(-\infty,-2]\cup[0,\infty)$

100. $P(x)=a_1x^2+2b_1x+c_1>0;\ D_1=4(b_1^2-a_1c_1)<0,\ a_1>0, c_1>0$

$Q(x)=a_2x^2+2b_2x+c_2>0;\ D_2=4(b_2^2-a_2c_2)<0,\ a_2>0, c_2>0$

$f(x)=a_1a_2x^2+b_1b_2x+c_1c_2$

$D=b_1^2b_2^2-4a_1a_2c_1c_2<0$

101. $x^2-2x+4=-3\cos(ax+b)$

$(x-1)^2+3=-3\cos(ax+b)$

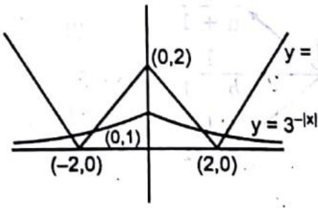
$\Rightarrow \quad x=1$ and $ax+b=\pi$

102. $\alpha+\beta=\alpha+\alpha\cdot r=4$

$\gamma+\delta=\alpha\cdot r^2+\alpha r^3=36\ \Rightarrow r=3,\alpha=1$

$A=3,\ B=243=3^5$

103.



104. We have $4x^2-16x+15<0\Rightarrow\frac{3}{2}<x<\frac{5}{2}\Rightarrow\cot\alpha=2$, the integral solution of the given inequality and $\sin\beta=\tan 45°=1$

$\therefore\quad \sin(\alpha+\beta)\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta=\frac{1}{1+\cot^2\alpha}-1=\frac{1}{1+4}-1=-\frac{4}{5}$

105. $f_1(x)=f_2(x)$

$\Rightarrow\ 2+\log_e x=x$

$\Rightarrow\ \log_e x=(x-2)$

Clearly graphs intersect once in (0,1).

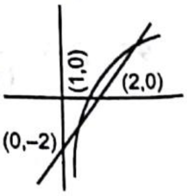
Now check

$\Rightarrow\ g(x)=2+\ln x-x$

$g(e)>0$

$g(e^2)<0$

$\Rightarrow$ one root between (e,e^2)



106. $x^4 - 3x^3 - 2x^2 - 3x + 1 = 0 \qquad \Rightarrow x^2 - 3x - 2 - \frac{3}{x} + \frac{1}{x^2} = 0$

$$\Rightarrow \left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) - 2 = 0 \Rightarrow \left(x + \frac{1}{x}\right)^2 - 3\left(x + \frac{1}{x}\right) - 4 = 0$$

$\Rightarrow \quad t^2 - 3t - 4 = 0 \quad (\text{where } x + \frac{1}{x} = t)$

$\Rightarrow \quad (t-4)(t+1) = 0 \quad \Rightarrow t = 4 \text{ or } t = -1$

$\Rightarrow \quad x + \frac{1}{x} = 4 \text{ or } x + \frac{1}{x} = -1$

Real solutions are from $x + \frac{1}{x} = 4 \quad \Rightarrow x^2 + 1 = 4x \Rightarrow x^2 - 4x + 1 = 0$

Hence, sum of roots = 4.

107. $f(x) = x^2 - (k+4) + k^2 - 12$

$f(4) = 16 - 4(k+4) + k^2 - 12 < 0$

$\Rightarrow \quad -2 < k < 6$

108. $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = k^2 - 2(k^2 + 2k - 4) = -k^2 - 4k + 8$

Maximum value = 12

109. $f(x) = a^x - x \ln a$

$f'(x) = (a^x - 1) \cdot \ln a$

110. As a, b and c are the roots of $x^3 + 2x^2 + 1 = 0$, we have

$$a + b + c = -2$$
$$ab + bc + ca = 0$$

Now, for finding the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, evaluating using first row, we get

$$\begin{aligned} a(bc - a^2) - b(b^2 - ac) + c(ab - c^2) &= abc - a^3 - b^3 + abc + abc - c^3 \\ &= 3abc - a^3 - b^3 - c^3 \\ &= -(a^3 + b^3 + c^3 - 3abc) \\ &= -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= -(-2)[(-2)^2 - 3(0)] = 8 \end{aligned}$$

111. $x^2 + px + q = 0,\ p, q \in R, q \neq 0 \quad \alpha, \beta$ real roots.

$g(x) = 0 \qquad \alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}$

$$\alpha+\beta+\frac{1}{\alpha}+\frac{1}{\beta}=\left(\alpha+\frac{1}{\alpha}\right)\left(\beta+\frac{1}{\beta}\right)$$

$$-p+\frac{-p}{q}=\alpha\beta+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+\frac{1}{\alpha\beta}$$

$$-p+\frac{-p}{q}=q+\frac{p^2-2q}{q}+\frac{1}{q}$$

$$-pq-p=q^2+p^2-2q+1$$

$$p^2+p(p+1)+q^2-2q+1=0$$

$$(q+1)^2-4(q^2-2q+1)\geq 0$$

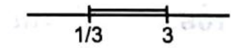
$$q^2+2q+1-4q^2+8q-4\geq 0$$

$$-3q^2+10q-3\geq 0$$

$$3q^2-2q-q+3\leq 0$$

$$3q(q-3)-(q-3)\leq 0$$

$$\left[\frac{1}{3},3\right]$$



112. $\ln(x^2+5x)=\ln(x+a+3)\ \Rightarrow\ x^2+5x=x+a+3>0$

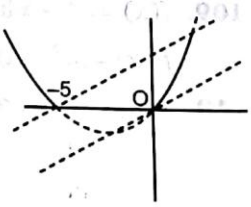
$$a+3>0$$

$$a>-3$$

$y=x+a+3\ \Rightarrow\ -5+a+3\leq 0$

$$a\leq 2$$

$$-3<a\leq 2$$



113. $f(x)=x^2+\dfrac{1}{x^2}-6x-\dfrac{6}{x}+2=\left(x+\dfrac{1}{x}\right)^2-6\left(x+\dfrac{1}{x}\right)$

Let $x+\dfrac{1}{x}=t$

$f(x)=t^2-6t\ \forall\ t\in(-\infty,-2]\cup[2,\infty)$

min. value $=-9$ at $t=3$

114. $x^3+2x^2+2x+c=(x^2+bx+b)\left(x+\dfrac{c}{b}\right)$

$\Rightarrow\ b+\dfrac{c}{b}=2$ and $b+c=2\Rightarrow b=c=1$

115. $\alpha\beta+\beta\gamma+\alpha\gamma=0\Rightarrow(\alpha\beta)^3+(\beta\gamma)^3+(\alpha\gamma)^3=3(\alpha\beta)(\alpha\gamma)(\beta\gamma)$

118. $\sum_{r=1}^{\infty}(\alpha^r+\beta^r)=(\alpha+\alpha^2+\alpha^3+\ldots)+(\beta+\beta^2+\beta^3+\ldots)$

$$=\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}$$

$$4x^2+2x-1=0 \begin{array}{l} \nearrow \alpha \\ \searrow \beta \end{array}$$

$$4\left(\frac{x}{1+x}\right)^2+2\left(\frac{x}{1+x}\right)-1=0 \Rightarrow 5x^2-1=0 \begin{array}{l} \nearrow \frac{\alpha}{1-\alpha} \\ \searrow \frac{\beta}{1-\beta} \end{array}$$

119. $\left(\frac{2011}{2014}\right)^x+\left(\frac{2012}{2014}\right)^x+\left(\frac{2013}{2014}\right)^x=1$

Let $f(x)=\left(\frac{2011}{2014}\right)^x+\left(\frac{2012}{2014}\right)^x+\left(\frac{2013}{2014}\right)^x \Rightarrow f(x)$ is a decreasing function for $x \in R$.

123.

$$x^2+ax+12=0 \quad \ldots(1)$$
$$x^2+bx+15=0 \quad \ldots(2) \qquad \text{Common roots}$$
$$x^2+(a+b)x+36=0 \quad \ldots(3)$$

(1) + (2) − (3)

$$\alpha^2=9 \Rightarrow \alpha=\pm 3$$

positive root $\alpha=3$

124. $e^{\sin x}=t$

$t^2-4t-1=0 \Rightarrow t=2\pm\sqrt{5} \Rightarrow e^{\sin x}=2\pm\sqrt{5}$ (Not possible)

125. Maximum value of $f(x)=3$

Minimum value of $f(x)=-1$

126. $f(1)=\lambda-2<0$

127. $2x^2+5x+7=0$ has non-real roots $\Rightarrow \frac{a}{2}=\frac{b}{5}=\frac{c}{7}$

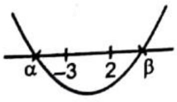
Min. value of $a+b+c=2+5+7=14$

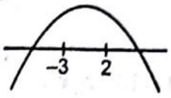
Max. value of $a+b+c=28+70+98=196$

128. Distance $=\sqrt{(x_B-x_A)^2+(y_B-y_A)^2}=\sqrt{(1-2t)^2+t^2}=\sqrt{5t^2-4t+1}$

Min. distance $=\frac{1}{\sqrt{5}}$ at $t=\frac{2}{5}$

129.

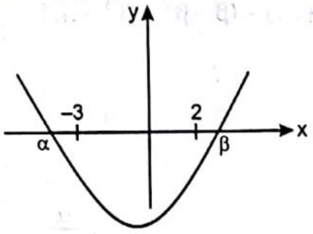




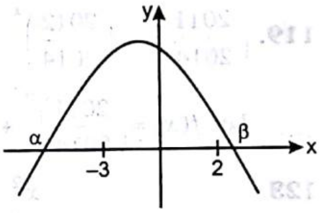
$af(1)<0$ and $af(-1)<0$; $f(-3)f(2)>0$

We have the equation $ax^2+bx+c=0$ has two roots α and β such that $\alpha<-3$ and $\beta>2$.

If $a > 0$, then we have the following graphical representation :



Then, for all $x \in [-3, 2]$, $f(x) < 0$, we have the following graphical representation :



This implies that

$$f(-1) < 0 \text{ and } f(1) < 0$$

$$\Rightarrow \quad a - b + c < 0 \text{ and } a + b + c < 0$$

$$\Rightarrow \quad a(a + |b| + c) < 0$$

If $a < 0$, then for all $x \in [-3, 2]$, $f(x) > 0$. This imply that

$$\Rightarrow \quad f(-1) > 0 \text{ and } f(1) > 0$$

$$\Rightarrow \quad a - b + c > 0 \text{ and } a + b + c > 0$$

$$\Rightarrow \quad a(a + |b| + c) < 0$$

130. Let $x^2 + 5x = t$

$$t^2 - 2t - 24 = 0 = (t - 6)(t + 4)$$

$$x^2 + 5x - 6 = 0 = (x + 6)(x - 1)$$

$$x^2 + 5x + 4 = 0 = (x + 4)(x + 1)$$

131. Case-1: $x \geq 2$

$$3(x - 2) - (1 - 5x) + 4(3x + 1) = 13 \Rightarrow x = \frac{4}{5} \quad \text{(Not possible)}$$

Case-2: $\frac{1}{5} \leq x < 2$

$$-3(x - 2) - (1 - 5x) + 4(3x + 1) = 13 \Rightarrow x = \frac{2}{7} \quad \text{(Possible)}$$

Case-3: $-\frac{1}{3} \leq x < \frac{1}{5}$

$$-3(x - 2) + (1 - 5x) + 4(3x + 1) = 13 \Rightarrow x = \frac{1}{2} \quad \text{(Not possible)}$$

Case-4: $x < -\frac{1}{3}$

$$-3(x - 2) + (1 - 5x) - 4(3x + 1) = 13 \Rightarrow x = -\frac{1}{2} \quad \text{(Possible)}$$

132. $\log_{\cos x} \sin x \ge 2 \Rightarrow \sin x \le \cos^2 x$

$\sin^2 x + \sin x - 1 \le 0$

$$0 < \sin x \le \frac{\sqrt{5}-1}{2} \qquad (\sin x > 0)$$

133. Minimum value $\frac{-D}{4} = -5 \Rightarrow D = 20$

$$|\alpha - \beta| = \frac{\sqrt{D}}{1} = \sqrt{20}$$

134. $|x-3| + |x+5| = 7x$

$2x + 2 = 7x \qquad x \ge 3$

$-(x-3) + (x+5) = 7x \qquad -5 < x < 3$

$-(x-3) - (x+5) = 7x \qquad x \le -5$

136. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ac) \Rightarrow ab + bc + ac = -4$

$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac) \Rightarrow abc = -4$

140. $x^2 - 3x + 4 < x^2 + 3x + 4$

$\Rightarrow \qquad x > 0$

142. $x^2 + 4x + 3 = 0 \begin{matrix} \nearrow \alpha \\ \searrow \beta \end{matrix}$

$\alpha = -3, \beta = -1$

143. $a^3 + b^3 + c^3 = 3abc$

$\Rightarrow \quad a + b + c = 0$

$\Rightarrow \quad ax^2 + bx + c = 0$ has one root $x = 1$

145. $x_1 + x_2 + x_1x_2 = a$

$x_1x_2 + x_1x_2(x_1 + x_2) = b$

$x_1^2x_2^2 = c \quad \Rightarrow b + c = x_1x_2(a+1)$

147. $(|x|-2)(|x|-1) = 0 \quad \Rightarrow x = \pm 1, \pm 2$

149. $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 4(1 - \sin 2\theta)^2 + 4\cos^2 2\theta$

$= 4(2 - 2\sin 2\theta)$

150. $\sin^2 x + \sin x = -b \ \forall\ x \in [0, \pi]$

$0 \le -b \le 2$

$-2 \le b \le 0$

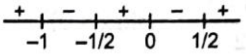
152. $x^2 + px - r = 0 = (x-\gamma)(x-\delta)$

$\alpha^2 + p\alpha - r = (\alpha - \gamma)(\alpha - \delta) = -q - r$

153. $2^{x+2} - 4^x \le 9 \cap 2^{x+2} - 4^x > 0;\ 2^x(4 - 2^x) > 0$

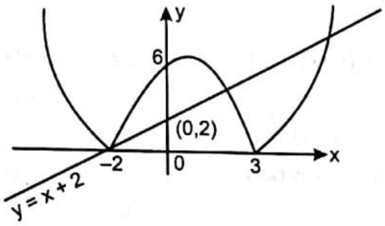
Exercise-2 : One or More than One Answer is/are Correct

1. $$\frac{2x-1}{x(x+1)(2x+1)} > 0$$



$$\Rightarrow \quad x \in (-\infty, -1) \cup \left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$$

3.



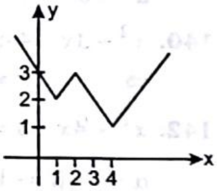
4. Apply $D \geq 0 \cap f(2) > 0 \cap f(-2) > 0 \cap -2 < \frac{a}{2} < 2$

5. $f(x) = x - 3 \qquad x > 4$

$= 5 - x \qquad 2 < x < 4$

$= x + 1 \qquad 1 < x < 2$

$= 3 - x \qquad x < 1$



6. $a - b = \frac{1}{b} - \frac{1}{a} \qquad \Rightarrow ab = 1 \qquad (\because a \neq b)$

$a - b = \frac{a}{b} \qquad \Rightarrow a - \frac{1}{a} = a^2 \quad \Rightarrow a^3 - a^2 + 1 = 0$

7. If $f(2+x) = f(2-x)$ and $D > 0$

Vertex of parabola is $\left(2, -\frac{D}{4a}\right)$ lies in IVth quadrant.

$f(0) > f(1) > f(2)$

8. If $f(2+x) = f(2-x)$ and $D < 0$

$f(-2) = 4a - 2b + c > 0$

If $\log_{f(2)} f(3)$ is not defined then $f(2) = 1$

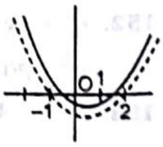
$\Rightarrow \quad f(x) \geq 1$

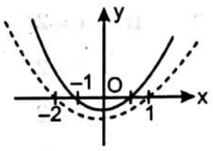
If $\frac{-b}{2a} = 2 \Rightarrow a$ and b are opposite sign.

9. **Case-I :** $f(-1) \geq 0 \cap f(1) < 0 \cap f(2) \geq 0$

$a \leq 0 \cap a < 0 \cap a \geq -\frac{3}{2}$

$\Rightarrow \qquad a \in \left[-\frac{3}{2}, 0\right)$





Case-II: $f(1) \geq 0 \cap f(-1) < 0 \cap f(-2) \geq 0$

$$a \geq 0 \cap a > 0 \cap a \leq \frac{3}{2}$$

$$\Rightarrow \quad a \in \left(0, \frac{3}{2}\right]$$

10. As expression taking minimum value

So, $a > 0$

$$\frac{-b}{2a} < 0; \quad \frac{-D}{4a} < 0$$

$\Rightarrow \quad a > 0, b > 0, D > 0$

11. $ax^2 + bx + c > 0 \; \forall \; x \in R$

$a > 0, D < 0$

$f(0) = c > 0$

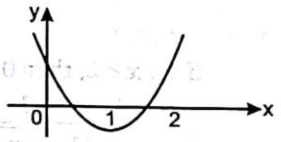
$f(-3) + f(-2) = 13a - 5b + 2c > 0$

$f(-3) + f(2) = 13a - b + 2c > 0$

12. $D > 0 \quad \Rightarrow k > \sqrt{5}$

$f(1) < 0 \quad \Rightarrow k > 3$

$f(2) > 0 \quad \Rightarrow k < \frac{21}{4}$



$$\Rightarrow \quad 3 < k < \frac{21}{4}$$

13.
$$x^2 + px + q = 0$$

Sum of the roots $= -13$

Product of the roots $= 30$

$$\Rightarrow \quad x^2 + 13x + 30 = 0 = (x + 10)(x + 3)$$

$\Rightarrow$ Correct roots are $x = -10, -3$

14. $x^2 - 3x + 2 > 0$

$(x - 2)(x - 1) > 0 \Rightarrow x \in (-\infty, 1) \cup (2, \infty)$

$x^2 - 3x - 4 \leq 0$

$(x - 4)(x + 1) \leq 0 \Rightarrow x \in [-1, 4]$

then $x \in [-1, 1) \cup (2, 4]$

15. $5^x + (2\sqrt{3})^{2x} - 169 \leq 0$

$5^x + 12^x - 169 \leq 0$

if $x = 2$ $\quad 5^2 + 12^2 = 169$

$x > 2$ $\quad 5^x + 12^x > 169$

$x < 2$ $\quad 5^x + 12^x < 169$

$\Rightarrow \quad x \in (-\infty, 2]$

16. $f(x) = x^2 + ax + b$

$D_1 : a^2 - 4b$

$g(x) = x^2 + cx + d$

$D_2 : c^2 - 4d$

$D_1 + D_2 = a^2 + c^2 - 4(b + d) = (a - c)^2 > 0 \Rightarrow$ atleast one of them is positive.

17. Let $x - 1 = t^2$

$$\frac{1}{\sqrt{x + 2\sqrt{x-1}}} + \frac{1}{\sqrt{x - 2\sqrt{x-1}}} = \frac{1}{\sqrt{t^2 + 2t + 1}} + \frac{1}{\sqrt{t^2 - 2t + 1}}$$

$$= \frac{1}{|t+1|} + \frac{1}{|t-1|} = \frac{1}{|1 + \sqrt{x-1}|} + \frac{1}{|\sqrt{x-1} - 1|}$$

If $1 < x < 2$, then $0 < \sqrt{x-1} < 1$

$$\frac{1}{|1 + \sqrt{x-1}|} + \frac{1}{|\sqrt{x-1} - 1|} = \frac{1}{1 + \sqrt{x-1}} + \frac{1}{1 - \sqrt{x-1}} = \frac{2}{2 - x}$$

If $x > 2$, then $\sqrt{x-1} > 1$

$$\frac{1}{|1 + \sqrt{x-1}|} + \frac{1}{|\sqrt{x-1} - 1|} = \frac{1}{\sqrt{x-1} + 1} + \frac{1}{\sqrt{x-1} - 1} = \frac{2\sqrt{x-1}}{x - 2}$$

18. $\log_{1/3}(x^2 + 2px + p^2 + 1) \geq 0$

$\Rightarrow \quad (x + p)^2 + 1 \leq 1 \;\Rightarrow (x + p)^2 \leq 0 \;\Rightarrow x = -p$

$kp^2 - kp - k^2 \leq 0 \;\forall\; k \in R$

$k^2 + (p - p^2) \geq 0 \;\forall\; k \in R$

$D \leq 0$

19. (a) $\quad \alpha + \beta = \alpha^2 + \beta^2$

and $\alpha\beta = \alpha^2\beta^2 \Rightarrow \alpha\beta(\alpha\beta - 1) = 0 \Rightarrow \alpha = 0$ or $\beta = 0$ or $\alpha\beta = 1$

(b) $\tan 2\theta + \tan 3\theta = \dfrac{\sin 5\theta}{\cos 2\theta \cos 3\theta} = 0 \Rightarrow \sin 5\theta = 0 \Rightarrow \theta = \dfrac{n\pi}{5}$

(c) $\dfrac{\left(\dfrac{2x_1}{x_2} + \dfrac{128x_3^2}{x_2^2} + \dfrac{x_2^3}{4x_1x_3^2}\right)}{3} \geq 4 \qquad (\because \text{AM} \geq \text{GM})$

(d) Equation of chord with mid-point (h, k) is $T = S_1$

$\Rightarrow \quad (h-1)x + (k-3)y + (h + 3k - h^2 - k^2) = 0$

If it is passes from $(0, 0)$.

Then, $h^2 + k^2 - h - 3k = 0$

20. $-2 < a < 2 \qquad \Rightarrow a^2 \in [0, 4)$

$x^2 - 4x - a^2 = 0 \Rightarrow x = 2 \pm \sqrt{4 + a^2}$

21. If $\alpha + 2\beta = 0$

$\Rightarrow \alpha\beta < 0 \Rightarrow -2\beta^2 < 0 \Rightarrow q < 0$

$\alpha + \beta = -\beta = p$

$\alpha\beta = -2\beta^2 = q \Rightarrow 2p^2 + q = 0$

22. $f(x) = (a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b) = 0 \Rightarrow f(1) = 0$

(a) if $a > b > c > 0$

$\Rightarrow \quad a + b > 2c$

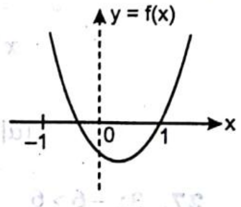
$f(0) = a + c - 2b < 0$

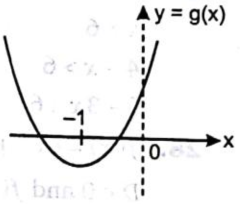
(c) $g(x) = ax^2 + 2bx + c = 0 \begin{matrix} \nearrow \alpha \\ \searrow \beta \end{matrix}$

$g(0) = c > 0$

$g(-1) = a - 2b + c < 0$

(d) $cx^2 + 2bx + a = 0 \begin{matrix} \nearrow 1/\alpha \\ \searrow 1/\beta \end{matrix}$





23. $f(x) = 4x^2 - 8ax + a$

$D = 48a^2 \geq 0$

(a) If $f(x)$ is non-negative $\forall\, x \in R$, then $a = 0$

(b) If $a < 0$, then $f(0) < 0$

(c) If $f(x) = 0$ has two distinct solutions in $(0, 1)$, then

$$f(0) > 0 \Rightarrow a > 0$$

$$f(1) > 0 \Rightarrow a < \frac{4}{7}$$

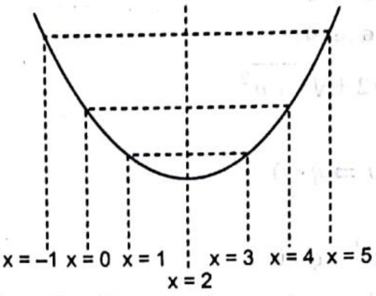
$$0 < \frac{-b}{2a} < 1 \Rightarrow 0 < a < 1$$

24. $ax^2 + bx + c = 0$ has no real roots, then $D < 0$

$$f\left(-\frac{1}{2}\right) = a - 2b + 4c > 0 \Rightarrow a > 0$$

$$\frac{4a+2b+c}{a+3b+9c}=\frac{f(2)}{f\left(\frac{1}{3}\right)}>0$$

25.



26. $ax^2+bx+c=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b\pm i\sqrt{4ac-b^2}}{2a}$$

$$|\alpha|=|\beta|=\sqrt{\left(\frac{-b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}=\sqrt{\frac{c}{a}}$$

27. $3x-6>6$ $\qquad x>3$
$x>6$ $\qquad 2\le x\le 3$
$4-x>6$ $\qquad 1\le x<2$
$6-3x>6$ $\qquad x<1$

28. $f(x)=ax^2+x+b-a$
$D<0$ and $f(1)=b+1>0$
$f(0)=b-a>0$
$f(1/2)=4b+2-3a>0$

29.

$$a^2+b^2=(a+b)^2-2ab=7 \qquad \text{...(1)}$$

$$a^3+b^3=(a+b)(7-ab)=10 \qquad \text{...(2)}$$

$\Rightarrow \quad (a+b)[21-(a+b)^2]=20$

Let $\qquad (a+b)=x$

$$x^3-21x+20=0$$

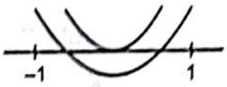
$$(x-1)(x+5)(x-4)=0$$

31. $\alpha+\beta+\gamma+\delta=0$

Root $=-\frac{1}{\delta},-\frac{1}{\gamma},-\frac{1}{\beta},-\frac{1}{\alpha}$

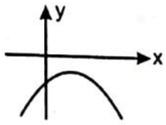
Put $x\rightarrow-\frac{1}{x}$

32. $D \geq 0 \ \cap\ f(-1) > 0 \ \cap\ f(1) > 0$



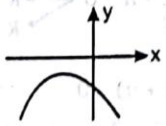
$-2 < K \leq \frac{1}{4}$

33. (a)



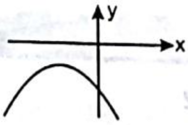
$a < 0$
$c < 0$
$b < 0$

(b)



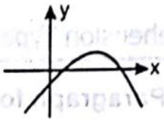
$a < 0$
$c < 0$
$b > 0$

(c)



$a < 0$
$c > 0$
$b > 0$

(d)



$a < 0$
$c < 0$
$b < 0$

34. (a) $f(1)f(-1) > 0$ (b) $f(1)f\left(-\frac{1}{2}\right) > 0$

(c) $f(-1)f(-2) > 0$ (d) $b^2 - 4ac < 0$

but a can be +ve or –ve.

35. $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha^2\beta^2} = \frac{b^2 - 2ac}{a^2}$$

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{(\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)}{(\alpha\beta)^3} = \frac{-b(b^2 - 3ac)}{c^3}$$

36. $\lambda = \sin^2 x + \sin x - 1$

$-1 < \lambda < 1$

37. $x^2 + 5x = x + a + 3 \ \forall\ x \in (-5, 0)$

$x^2 + 4x - 3 = a \ \forall\ x \in (-5, 0)$

$\Rightarrow \quad a \in (-3, 2]$

39. $x^2 - 2ax - a^2 = 0 \qquad x > a$

$\Rightarrow \quad x = a(1 \pm \sqrt{2})$

$x^2 + 2ax - 5a^2 = 0 \qquad x < a$

43. $(\alpha + \beta) + (\gamma + \delta) = 12 \quad \Rightarrow \alpha + \beta = \gamma + \delta = 6$ ...(1)

$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = 54 \quad \Rightarrow \quad \alpha\beta + \gamma\delta = 9$ ...(2)

$(\alpha\beta)(\gamma\delta) = 14$ ...(3)

$\Rightarrow \quad \alpha\beta = 7,\ \gamma\delta = 2$

44. $$l\left(\frac{K^3}{K-1}\right)+m\left(\frac{K^2-3}{K-1}\right)+n=0 \begin{array}{l}\nearrow K=a\\ \rightarrow K=b\\ \searrow K=c\end{array}$$

$lK^3 + mK^2 + nK - (3m+n) = 0$

Exercise-3 : Comprehension Type Problems

Paragraph for Question Nos. 1 to 2

1. $f(-1-x) = f(-1+x)\ \forall\ x \in R$

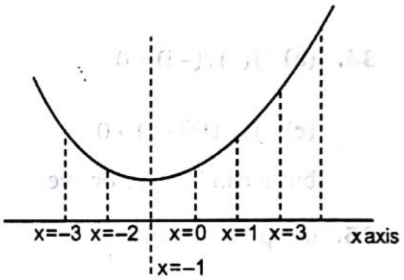
$\Rightarrow$ graph of $f(x)$ is symmetric about $x = -1$.

$$-\frac{b}{2a} = -1 \Rightarrow b = 2a$$

$$\alpha = f(-2) = 4a - 2b + c$$

$$\beta = f(3) = 9a + 3b + c$$

$$\gamma = f(-3) = 9a - 3b + c$$



Using graph $f(3) > f(-3) > f(-2) \Rightarrow \beta > \gamma > \alpha$

2. $p = b - 4a = -2a < 0$

$q = 2a + b = 4a > 0$

$\Rightarrow \qquad p \times q < 0$

Paragraph for Question Nos. 3 to 4

Sol. $(k+1)x^2 - (20k+14)x + 91k + 40 = 0$

$\left.\begin{array}{l} f(4) = 27k > 0 \\ f(7) = -9 < 0 \end{array}\right] \rightarrow$ One root is lie (4, 7)

$\left.\begin{array}{l} f(10) = -9k < 0 \\ f(13) = 27 > 0 \end{array}\right] \rightarrow$ Other root is lie (10, 13)

Paragraph for Question Nos. 5 to 7

5. $f(x) = x^2 + bx + c\ \forall\ x \in R$

Least value at $\frac{-b}{2} = -1 \Rightarrow b = 2$

Graph of $f(x)$ cuts y-axis, when $x = 0$

$\Rightarrow \qquad c = 2$

$\Rightarrow \quad f(x) = x^2 + 2x + 2$

Least value of $f(x) = 1$

6. $f(-2) + f(0) + f(1) = 9$

7. $a \in (1, \infty)$

Paragraph for Question Nos. 8 to 9

Sol. $(\log_2 x)^2 - 4(\log_2 x) - m^2 - 2m - 13 = 0$

8. $D > 0 \Rightarrow m^2 + 2m + 17 > 0 \ \forall\ m \in R$

9. $m^2 + 2m - (\log_2 x)^2 + 4(\log_2 x) + 13 = 0$

$D \geq 0$

$\Rightarrow \quad (\log_2 x - 6)(\log_2 x + 2) \geq 0 \Rightarrow x \in \left(0, \frac{1}{4}\right] \cup [64, \infty)$

Paragraph for Question Nos. 10 to 11

Sol. $x^4 - 2x^3 - 3x^2 + 4x - 1 = 0$ has four roots $a, \frac{1}{a}, b, \frac{-1}{b}$

$$\left(a + \frac{1}{a}\right) + \left(b - \frac{1}{b}\right) = 2 \quad \ldots(1)$$

$$\left(b - \frac{1}{b}\right) - \left(a + \frac{1}{a}\right) = -4 \quad \ldots(2)$$

Paragraph for Question Nos. 12 to 14

Sol. $f(x) - (6 - x) = 0 = (x-1)(x-2)(x-3)(x-4)(x-5)$

$f(x) = (x-1)(x-2)(x-3)(x-4)(x-5) + (6 - x)$

Paragraph for Question Nos. 15 to 16

Sol. $x^3 - x^2(1 + \sin\theta + \cos\theta) + x(\sin\theta + \cos\theta + \sin\theta\cos\theta) - \sin\theta\cos\theta = 0$

$\Rightarrow$ Roots are $1, \sin\theta, \cos\theta$.

Paragraph for Question Nos. 17 to 18

Sol. $2[1 + P(x)] = P(x-1) + P(x+1)$

$2 + 2[ax^2 + bx + c] = a(x-1)^2 + b(x-1) + c + a(x+1)^2 + b(x+1) + c$

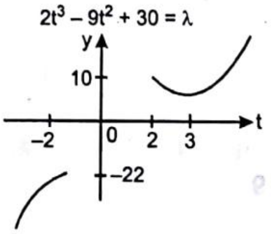
$\Rightarrow \quad a = 1$

$P(0) = c = 8$

$P(2) = 4a + 2b + c = 32 \Rightarrow b = 10$

Paragraph for Question Nos. 19 to 21

Sol.



Paragraph for Question Nos. 22 to 23

22. $D > 0$

$(2t-1)^2 - 4t(5t-1) > 0$

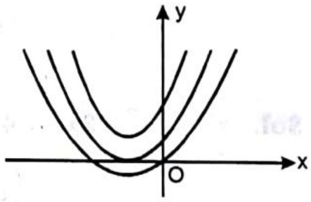
$16t^2 - 1 < 0 \Rightarrow \frac{-1}{4} < t < \frac{1}{4} \qquad (t \neq 0)$

23. $t > 0$

Case-I: $D < 0$

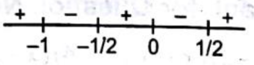
$(2t-1)^2 - 4t(5t-1) < 0 \Rightarrow t > \frac{1}{4} \quad (\because t > 0)$

Case-II: $D \geq 0 \cap f(0) \geq 0 \cap \frac{-b}{2a} \leq 0 \Rightarrow \frac{1}{5} \leq t \leq \frac{1}{4}$



Exercise-4 : Matching Type Problems

1. (A) $\frac{(2x-1)}{x(2x+1)(x+1)} > 0$



(B) $3x^2 + 2(a^2+1)x + (a^2 - 3a + 2) = 0$

$\alpha\beta < 0$

$a^2 - 3a + 2 < 0 \Rightarrow (a-1)(a-2) < 0$

(C) $\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1$

Let $x - 1 = t^2$; $|t-2| + |t-3| = 1$

(D) $\text{A.M.} = \frac{\alpha+\beta+\gamma+\delta}{4} = 2$

$\text{G.M.} = (\alpha\beta\gamma\delta)^{1/4} = 2$

$\text{A.M.} = \text{G.M.} \Rightarrow \alpha = \beta = \gamma = \delta = 2$

3. (A) $x^4 - 8x^2 - 9 = 0$

$(x^2 - 9)(x^2 + 1) = 0 \quad \Rightarrow x = 3, -3$

(B) $x^{2/3} + x^{1/3} - 2 = 0$

$(x^{1/3} + 2)(x^{1/3} - 1) = 0 \quad \Rightarrow x = -8, 1$

(C) $(\sqrt{3x+1})^2 = (\sqrt{x} - 1)^2$

$\Rightarrow \quad 3x + 1 = x + 1 - 2\sqrt{x} \quad \Rightarrow 2x = -2\sqrt{x}$ (Not possible)

(D) $(3^x - 9)(3^x - 1) = 0 \quad \Rightarrow x = 0, 2$

4. (A) $\therefore \quad (a + b) = -a$ & $ab = b \Rightarrow (1, -2)$ and $(0, 0)$

(B) $P = \overline{O}, \quad Q = 8\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = 1$

(C) $ar^6 = \sqrt{2}$

Product $= (\sqrt{2})^{11} = 2^{11/2}$

$\therefore \quad m = 11$

$n = 4$

(D) $x = y = 3$

$\because \quad (x - y)^2 + (y - 3)^2 = 0$

$\therefore \quad 5x - 4y = 3$

Exercise-5 : Subjective Type Problems

1. $f\left(\cos\frac{\pi}{7}\right) = \sin\frac{\pi}{7}\sin\frac{3\pi}{7} + \sin\frac{3\pi}{7}\sin\frac{5\pi}{7} + \sin\frac{\pi}{7}\sin\frac{5\pi}{7}$

$= 2\cos^2\frac{\pi}{7} + \cos\frac{\pi}{7} - 1$

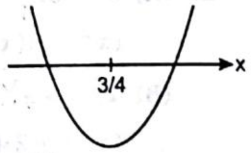
$\Rightarrow \quad f(x) = 2x^2 + x - 1$

2. $(r - a)(r - b)(r - c)(r - d) = (-1) \times (-3) \times (1) \times (3)$

$\Rightarrow \quad (r - a) + (r - b) + (r - c) + (r - d) = 0$

3. Let $x^2 + x + 1 = t \quad \forall\, t \in \left[\frac{3}{4}, \infty\right)$

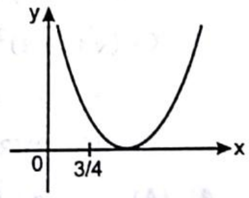
$t^2 - (m - 3)t + m = 0$



Case-I : $f\left(\frac{3}{4}\right)<0$

$\frac{9}{16}-\frac{3}{4}(m-3)+m<0$

$\Rightarrow$ $m<\frac{-45}{4}$



Case-II : $D=0 \quad \Rightarrow m=1,9$

$\frac{-b}{2a}>\frac{3}{4} \Rightarrow m>\frac{9}{2}$

There is one positive integral value of $m=9$.

4. $t^2-(m-3)t+m=0$

$t\in[3/4,\infty)$ has four distinct real roots, then

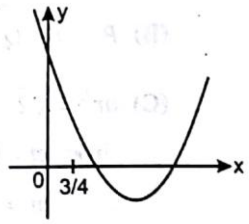
$D>0$

$\Rightarrow \; m^2-10m+9>0$

$\Rightarrow \; m\in(-\infty,1)\cup(9,\infty)$

$\frac{-b}{2a}>\frac{3}{4} \quad \Rightarrow m>\frac{9}{2}$

$f\left(\frac{3}{4}\right)>0 \Rightarrow m>\frac{-45}{4} \quad \Rightarrow m\in(9,\infty)$



5. $f(t)=(m^2-12)t^2-8t-4=0 \qquad (t\geq 0)$

$f(0)=-4<0$

$m^2-12\leq 0 \;\Rightarrow m\in[-2\sqrt{3},2\sqrt{3}]$

Case-I : $D<0$

$\Rightarrow$ $m^2-8<0 \;\Rightarrow m\in(-2\sqrt{2},2\sqrt{2})$

Case-II : $D\geq 0 \;\Rightarrow m\in(-\infty,-2\sqrt{2}]\cup[2\sqrt{2},\infty)$

$\frac{-b}{2a}=\frac{4}{m^2-12}<0 \;\Rightarrow m\in(-2\sqrt{3},2\sqrt{3})$

$\Rightarrow$ $m\in[-2\sqrt{3},2\sqrt{3}]$

6. $(e^x-2)\left[\sin\left(x+\frac{\pi}{4}\right)\right](x-\ln 2)(\sin x-\cos x)<0$

$\frac{1}{\sqrt{2}}(x-\ln 2)\cdot(\sin x+\cos x)(x-\ln 2)(\sin x-\cos x)<0=\frac{1}{\sqrt{2}}(x-\ln 2)^2(\sin^2 x-\cos^2 x)<0$

$\Rightarrow \; \cos 2x>0, \;\; x\neq \ln 2$

$$x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right] - \{\ln 2\}$$

Least positive integral value is 3.

7. $x^2 + 17x + 71 = \lambda^2 \quad \Rightarrow \lambda \in Z$

$x^2 + 17x + (71 - \lambda^2) = 0$

$$D = \text{perfect square} = m^2 \text{ (say)}$$

$$m^2 = 289 - 4(71 - \lambda^2)$$

$(m - 2\lambda)(m + 2\lambda) = 1 \times 5$

$\Rightarrow \quad m - 2\lambda = 1$

$m + 2\lambda = 5$

8. $P(x) = (x^4 - x^3 - x^2 - 1)(x^2 + 1) + (x^2 - x + 1)$

$P(\alpha) + P(\beta) + P(\gamma) + P(\delta) = (\alpha^2 - \alpha + 1) + (\beta^2 - \beta + 1) + (\gamma^2 - \gamma + 1) + (\delta^2 - \delta + 1) = 6$

9. If $-\frac{a}{2} \le 1$

$f(x)_{\max} = f(4) \Rightarrow 4a + 18 = 6 \Rightarrow a = -3$ (Not possible)

if $-\frac{a}{2} \ge 1$

$f(x)_{\max} = f(-2) \Rightarrow a = 0$ (Not possible)

There is no real value of 'a'.

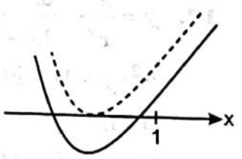
10. $x^2 - 8x - (n^2 - 10n) = 0$

$D < 0 \quad \Rightarrow n^2 - 10n + 16 < 0$

$(n - 8)(n - 2) < 0$

$\Rightarrow 2 < n < 8$ and $n \ne 10$

11. $x^2 + 2(m - 1)x + (m + 5) > 0 \; \forall \; (x > 1)$



Case-I : $D < 0$

$m^2 - 3m - 4 < 0 \Rightarrow -1 < m < 4$

Case-II : $D \ge 0$

$\Rightarrow \quad m \in (-\infty, -1] \cup [4, \infty)$

$f(1) \ge 0 \Rightarrow m \ge -\frac{4}{3}$

$\frac{-b}{2a} < 1 \quad \Rightarrow m > 0$

$\Rightarrow \quad m \in (-1, \infty)$

12. $ax^4 + bx^3 - x^2 + 2x + 3 = (x+2)(x-1)\,Q(x) + (4x+3)$

Put $x = 1 \qquad a + b = 3$

$x = -2 \qquad b = 2a$

13. $D > 0 \quad \cap \quad \dfrac{-b}{2a} > 4 \quad \cap \quad f(4) \geq 0$

$k - 1 > 0 \quad \cap \quad 4k > 4 \quad \cap \quad k^2 - 3k + 2 \geq 0$

$k > 0 \quad \cap \quad k > 1 \quad \cap \quad (k-2)(k-1) \geq 0$

$\Rightarrow \quad k \geq 2$

14. $x^2 - 3x + 2 = (x-1)(x-2)$

If $(x-1)$ is a factor of $x^4 - px^2 + q = 0$. Then

$$p - q = 1 \qquad \ldots(1)$$

If $(x-2)$ is a factor of $x^4 - px^2 + q = 0$. Then

$$4p - q = 16 \qquad \ldots(2)$$

$\Rightarrow \qquad p = 5, q = 4$

$\Rightarrow \qquad p + q = 9$

15. $x^2 + 2xy + ky^2 + 2x + k = 0$

if it can be resolved into two linear factors, then

$$abc + 2fgh - bg^2 - af^2 - ch^2 = 0$$

$$k^2 - k - k = 0$$

$$k = 0, 2$$

16. $(a+1)x^2 + 2 = ax + 3$ has exactly one solution.

$\Rightarrow \quad D = 0$

$a^2 + 4(a+1) = 0$

$(a+2)^2 = 0 \quad \Rightarrow a = -2 \quad \Rightarrow a^2 = 4$

17. $y = \dfrac{x^2 - ax + 1}{x^2 - 3x + 2}$

$x^2(y-1) - x(3y - a) + 2y - 1 = 0 \;\forall\; x \in R$

$D \geq 0$

$(3y - a)^2 - 4(y-1)(2y-1) \geq 0 \;\forall\; y \in R$

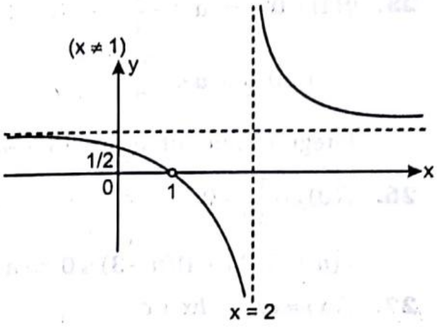
$y^2 - 6y(a-2) + a^2 - 4 \geq 0 \;\forall\; y \in R$

$D \leq 0$

$36(a-2)^2 - 4(a^2 - 4) \leq 0$

$(a-2)(2a-5) \leq 0$

$$2 \le a \le \frac{5}{2}$$



$\Rightarrow$ Integral value of $a = 2$

At $a = 2$

$$f(x) = \frac{x^2 - 2x + 1}{x^2 - 3x + 2} = \frac{(x-1)^2}{(x-2)(x-1)}$$

$$f(x) = \frac{x-1}{x-2} (x \ne 1)$$

Range $R - \{0, 1\}$

$\Rightarrow$ No integral values of 'a' for which range is R.

18. $x^{100} = (x^2 - 3x + 2) \cdot Q(x) + (ax + b)$

at $x = 1 \quad \Rightarrow \quad a + b = 1$ ...(1)

at $x = 2 \quad \Rightarrow \quad 2a + b = 2^{100}$ ...(2)

$\Rightarrow \quad a = 2^{100} - 1,\ b = 2 - 2^{100}$

Remainder $= (2^{100} - 1)\,x + 2(1 - 2^{99})$

$\Rightarrow \quad k = 99$

19. $x = 7^{1/3} + 7^{2/3}$

$x^3 = 7 + 49 + 3 \times 7(x) \quad \Rightarrow \quad x^3 - 21x - 56 = 0$

Product of all roots $= 56$

21. Clearly $P(x)$ is a second degree polynomial.

$\therefore \quad P(x) = ax^2 + bx + c$

$$P'(x) = 2ax + b$$

$$P(x) - P'(x) = ax^2 + (b - 2a)\,x + c - b = x^2 + 2x + 1$$

$$a = 1, b - 2a = 2, c - b = 1$$

$$a = 1, b = 4, c = 5$$

$$P(x) = x^2 + 4x + 5$$

$$P(-1) = 1 - 4 + 5 = 6 - 4 = 2$$

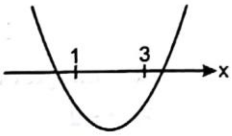
23. Let $x^2 = t$

$t^2 + kt + k = 0$

$D > 0 \quad \Rightarrow k \in (-\infty, 0) \cup (4, \infty)$

$f(0) < 0 \quad \Rightarrow k < 0$

25. $f(1) < 0 \Rightarrow a < -\frac{4}{3}$



$f(3) < 0 \Rightarrow a < -\frac{8}{7}$

Integral values of 'a' are $-5, -4, -3, -2$.

26. $f(0)f\left(\frac{\pi}{2}\right) \le 0$

$-(n+1)(2n+1)(n-3) \le 0 \Rightarrow n \in [3, \infty)$

27. $f(x) = ax^2 + bx + c \qquad a, b, c \in I$

$ax^2 + bx + c = a(x-\alpha)(x-\beta) + p \qquad \alpha, \beta \in I$

$ax^2 + bx + c - 2p = a(x-\alpha)(x-\beta) - p = 0$

Not possible for integral values of x.

28. $9x^2 + 2x(y-46) + y^2 - 20y + 244 = 0$

$D \ge 0 \quad \Rightarrow y^2 - 11y + 10 \le 0$

$(y-1)(y-10) \le 0 \Rightarrow 1 \le y \le 10$

$y^2 + 2y(x-10) + 9x^2 - 92x + 244 = 0$

$D \ge 0 \quad \Rightarrow x^2 - 9x + 18 \le 0$

$(x-3)(x-6) \le 0 \Rightarrow 3 \le x \le 6$

29. $a + b = 3$ and $a^3 + b^3 = 7 \Rightarrow a^3 + (3-a)^3 = 7 \Rightarrow 9a^2 - 27a + 20 = 0$

Sum of distinct values of 'a' is 3.

30. $(y^2-3)^2 + (x-4)^2 = 1$

$\Rightarrow \quad x = 4 + \cos\theta, \; y^2 = 3 + \sin\theta$

$M = 36, m = 1$

31. $x_1 + x_2 + x_1x_2 = a$

$x_1x_2 + x_1x_2(x_1 + x_2) = b$

$x_1^2x_2^2 = c$

If $\quad b + c = 2(a+1) \Rightarrow x_1x_2 = 2$

32. $x^3 + 3x^2 + 4x + 5 = 0 \quad \Rightarrow x = \alpha$ is root

$x^3 - 3x^2 + 4x - 5 = 0 \quad \Rightarrow x = \beta$ is root

$\Rightarrow \qquad \alpha + \beta = 0$

33. 5

$(1)-(2) \qquad\qquad (2)-(3)$

$(x-z)(x+y+z) = 1$ and $(y-x)(x+y+z) = -2$

Divide $z = \frac{x+y}{2}$

$$y^2 + y\left(\frac{x+y}{2}\right) + \left(\frac{x+y}{2}\right)^2 = 1 \text{ and } x^2 - 2xy - 5y^2 = 0$$

$$x = (1+\sqrt{6})y$$

$y^2 = \frac{2}{9+3\sqrt{6}}$ put values

34. $\frac{4(1-a-b)-(a-b)^2}{4} > \frac{4(1+a+b)-(a+b)^2}{-4}$

$\Rightarrow \quad 8 > (a+b)^2 + (a-b)^2 \Rightarrow a^2 + b^2 < 4$

35. $\sqrt[3]{20x + \sqrt[3]{20x+13}} = 13$

$$\sqrt[3]{20x + \sqrt[3]{20x + \sqrt[3]{20x + \sqrt[3]{20x + \sqrt[3]{20x + \ldots\infty}}}}} = 13$$

$\Rightarrow \quad \sqrt[3]{20x+13} = 13 \quad \Rightarrow 20x = 2197 - 13$

$\Rightarrow \quad x = \frac{2184}{20} = \frac{546}{5}$

36. Let $f(x) = x^2 - 2(a+1)x + a(a-1)$

$f(1-a) < 0 \quad \cap \quad f(1+a) < 0$

$4a^2 - 3a - 1 < 0 \quad \cap \quad 3a + 1 > 0$

$-\frac{1}{4} < a < 1 \quad \cap \quad a > -\frac{1}{3}$

$\Rightarrow \quad a \in \left(-\frac{1}{4}, 1\right)$

37. $(x-8)(x-2) < 0$

$\Rightarrow \quad 2 < x < 8$

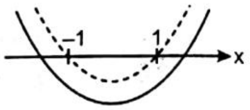
38. $\sin\theta + \cos\theta = \frac{-b}{a}$

$\sin\theta \cdot \cos\theta = \frac{c}{a}$

$(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta = \frac{b^2}{a^2} = 1 + \frac{2c}{a}$

$\Rightarrow \quad \frac{b^2 - a^2}{a^2} = \frac{2c}{a}$

39. $\cos^2 x + (1-a)\cos x - a^2 \le 0 \ \forall\ x \in R$



Let $\cos = t \ \forall\ t \in [-1,1]$

$t^2 + (1-a)t - a^2 \le 0 \ \forall\ t \in [-1,1]$

$f(-1) \le 0$

$\Rightarrow \quad a^2 - a \ge 0$

$\Rightarrow \quad a \in (-\infty, 0] \cup [1, \infty)$

$f(1) \le 0$

$\Rightarrow \quad a^2 + a - 2 \ge 0$

$(a+2)(a-1) \ge 0 \ \Rightarrow\ a \in (-\infty, -2] \cup [1, \infty)$

40. $2x^2 - 35x + 2 = 0 \genfrac{}{}{0pt}{}{\nearrow \alpha}{\searrow \beta}$

$2\alpha - 35 = -\dfrac{2}{\alpha}$ and $2\beta - 35 = -\dfrac{2}{\beta}$

42. $xF(x) - 1 = k(x-1)(x-2)(x-3)\ldots(x-9)$

$\Rightarrow \quad F(x) = \dfrac{k(x-1)(x-2)(x-3)\ldots(x-9)+1}{x}$

Constant term $= k(-9!) + 1 = 0$

$\Rightarrow \quad k = \dfrac{1}{9!}$

44. $\cos A + \cos B + \cos C = -a$

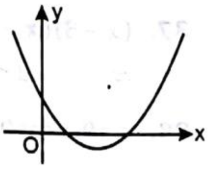
$\cos A \cos B + \cos B \cos C + \cos A \cos C = b$

$\cos A \cos B \cos C = -c$

$a^2 - 2b - 2c = \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C$

$= 1$

45. $k > 0$



$D > 0 \cap \dfrac{-b}{2a} > 0$

$k^2 - 10k + 9 > 0 \cap \dfrac{k-3}{k} < 0$

$k \in (-\infty, 1) \cup (9, \infty) \cap k \in (0, 3) \Rightarrow k \in (0, 1)$

❑❑❑

Solution to Chapter 9 till end (Chapter 26) is in part 2